Mathematics and Statistics (Arts and Science)

HSC Board Exam February–March 2027 – Practice Paper No. 2

Model Answer Paper

Time: 3 hours **Max. Marks: 80**

Model answers show one correct method; any other correct method gets full marks.

SECTION A – Q.1 Select and write the correct answer for the following multiple choice type of questions. **[16]**

(i) Which of the following statement patterns is a tautology?

(a) $q \rightarrow (p \wedge q)$ (b) $(p \wedge \sim q) \rightarrow p$

(c) $(p \vee q) \rightarrow p$ (d) $p \wedge (p \rightarrow \sim p)$

Answer (2 marks)

(b) $(p \wedge \sim q) \rightarrow p$

If $p \wedge \sim q$ is T, then p is T, so the implication is T. If $p \wedge \sim q$ is F, the implication is T anyway. So it is T in every row: a tautology.

(a) and (c) are F when p is F, q is T (for (c): $p \vee q$ is T, p is F). (d) is a contradiction.

Marking: Correct alternative with its alphabet – 2 marks

Examiner tip: Check each option in the row where it is most likely to fail: an implication fails only when its antecedent is T and its consequent is F.

(ii) If $A = \begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix}$ and the elementary transformation $R_2 \rightarrow R_2 - 2R_1$ is applied to A, then the second row of the new matrix is .....

(a) [0 5] (b) [0 1]

(c) [8 1] (d) [0 −5]

Answer (2 marks)

(a) [0 5]

New $R_2 = [4 - 2(2)\ 3 - 2(-1)] = [0\ 5]$; R_1 is unchanged, so the new matrix is $\begin{bmatrix} 2 & -1 \\ 0 & 5 \end{bmatrix}$.

Marking: Correct alternative with its alphabet – 2 marks

Examiner tip: Only the row being replaced changes. $3 - 2(-1) = 5$, not 1: watch the sign.

(iii) The value of $\cos[2 \sin^{-1}(1/3)]$ is .....

(a) $4\sqrt{2}/9$ (b) $-7/9$

(c) 7/9 (d) 2/3

Answer (2 marks)

(c) 7/9

Let $\sin^{-1}(1/3) = \alpha$, so $\sin \alpha = 1/3$.

$\cos 2\alpha = 1 - 2\sin^2\alpha = 1 - 2(1/9) = 7/9$. [(a) $4\sqrt{2}/9$ is $\sin 2\alpha$.]

Marking: Correct alternative with its alphabet – 2 marks

Examiner tip: Use $\cos 2\alpha = 1 - 2\sin^2\alpha$ directly; there is no need to find $\cos \alpha$.

(iv) The sum of the slopes of the lines represented by $3x^2 + 8xy - 5y^2 = 0$ is .....

(a) 8/5 (b) −8/5

(c) 3/5 (d) −3/5

Answer (2 marks)

(a) 8/5

For $ax^2 + 2hxy + by^2 = 0$, $m_1 + m_2 = -2h/b$.

Here $a = 3$, $2h = 8$, $b = -5$, so $m_1 + m_2 = -8/(-5) = 8/5$.

Marking: Correct alternative with its alphabet – 2 marks

Examiner tip: The formula is $-2h/b$ (divide by b, the coefficient of y^2), not $-2h/a$.

(v) If the vectors $\bar{a} = 2\hat{i} + \lambda\hat{j} + \hat{k}$ and $\bar{b} = \hat{i} - 2\hat{j} + 3\hat{k}$ are perpendicular, then λ = ...

(a) −5/2 (b) 3/2

(c) 5/2 (d) 1

Answer (2 marks)

(c) 5/2

$\bar{a} \perp \bar{b} \Rightarrow \bar{a} \cdot \bar{b} = 0 \Rightarrow 2(1) + \lambda(-2) + 1(3) = 0 \Rightarrow 5 - 2\lambda = 0 \Rightarrow \lambda = 5/2$.

Marking: Correct alternative with its alphabet – 2 marks

(vi) The point on the curve $y = x^2 - 6x + 11$ at which the tangent is parallel to the X-axis is .....

(a) (2, 3) (b) (3, 2)

(c) (−3, 38) (d) (3, −2)

Answer (2 marks)

(b) (3, 2)

$dy/dx = 2x - 6$. Tangent parallel to the X-axis $\Rightarrow dy/dx = 0 \Rightarrow x = 3$.

$y = 9 - 18 + 11 = 2$. So the point is (3, 2).

Marking: Correct alternative with its alphabet – 2 marks

(vii) $\int \cot x \cdot \log(\sin x)\, dx$ = ...

(a) $(1/2)[\log(\sin x)]^2 + c$ (b) $\log[\log(\sin x)] + c$

(c) $[\log(\sin x)]^2 + c$ (d) $(1/2)\log(\sin x) + c$

Answer (2 marks)

(a) $(1/2)[\log(\sin x)]^2 + c$

Put $\log(\sin x) = t \Rightarrow (\cos x/\sin x)\, dx = dt \Rightarrow \cot x\, dx = dt$.

$\int t\, dt = t^2/2 + c = (1/2)[\log(\sin x)]^2 + c$.

Marking: Correct alternative with its alphabet – 2 marks

(viii) The p.m.f. of a discrete random variable X is $P(X = x) = kx^2$ for x = 1, 2, 3 and 0 otherwise. Then $P(X \geq 2)$ = ...

(a) 1/14 (b) 9/14

(c) 2/7 (d) 13/14

Answer (2 marks)

(d) 13/14

$k(1 + 4 + 9) = 1 \Rightarrow k = 1/14$.

$P(X \geq 2) = P(2) + P(3) = 4/14 + 9/14 = 13/14$.

Marking: Correct alternative with its alphabet – 2 marks

Examiner tip: First find k from $\Sigma P(x) = 1$; only then add the required probabilities.

Q.2 Answer the following questions. **[4]**

(i) Find the principal solutions of $2\cos\theta + \sqrt{3} = 0$.

Answer (1 mark)
$\cos\theta = -\sqrt{3}/2$. $\cos\theta$ is negative in the 2nd and 3rd quadrants, and $\cos(\pi/6) = \sqrt{3}/2$.
$\theta = \pi - \pi/6 = 5\pi/6$ and $\theta = \pi + \pi/6 = 7\pi/6$.
$\therefore$ Principal solutions: $\boldsymbol{\theta = 5\pi/6, 7\pi/6}$.

(ii) Write the vector equation of the line passing through the point $(2, -1, 3)$ and parallel to the vector $4\hat{i} - \hat{j} + 2\hat{k}$.

Answer (1 mark)
$\bar{a} = 2\hat{i} - \hat{j} + 3\hat{k}$, $\bar{b} = 4\hat{i} - \hat{j} + 2\hat{k}$. The line is $\bar{r} = \bar{a} + \lambda\bar{b}$:
$\boldsymbol{\bar{r} = (2\hat{i} - \hat{j} + 3\hat{k}) + \lambda(4\hat{i} - \hat{j} + 2\hat{k})}$, λ a scalar.

(iii) Find the order and degree of the differential equation $(d^2y/dx^2)^3 + 4(dy/dx)^5 - 7y = \sin x$.

Answer (1 mark)
The highest order derivative is d^2y/dx^2, so the order is 2. Its power is 3 (the equation is a polynomial in the derivatives).
$\therefore$ **Order = 2, degree = 3**.

(iv) If $X \sim B(12, 1/4)$, find E(X) and Var(X).

Answer (1 mark)
$n = 12$, $p = 1/4$, $q = 3/4$.
$E(X) = np = \mathbf{3}$; $Var(X) = npq = 12 \times (1/4) \times (3/4) = \mathbf{9/4}$.

SECTION B Attempt any EIGHT of the following questions (Q.3 to Q.14, 2 marks each). **[16]**

Q.3 Using the rules of negation, write the negation of $p \rightarrow (q \wedge \sim r)$ and simplify it.

Answer (2 marks)
Negation $= \sim[p \rightarrow (q \wedge \sim r)]$
$\equiv p \wedge \sim(q \wedge \sim r)$ ..... $(\sim(p \rightarrow q) \equiv p \wedge \sim q)$
$\equiv p \wedge [\sim q \vee \sim(\sim r)]$ ..... (De Morgan's law)
$\equiv \mathbf{p \wedge (\sim q \vee r)}$ ..... $(\sim(\sim r) \equiv r)$
$[\equiv p \wedge (q \rightarrow r)$ ..... (Conditional law)]
Marking: Negation of the implication: $p \wedge \sim(q \wedge \sim r)$ – 1 mark · De Morgan's law and simplified form $p \wedge (\sim q \vee r)$ – 1 mark
Examiner tip: The negation of $p \rightarrow q$ is $p \wedge \sim q$ (not $\sim p \rightarrow \sim q$). Write the law used at each step.

Q.4 Find the inverse of $A = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$ by elementary row transformations.

Answer (2 marks)
Consider $AA^{-1} = I$: $\begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
$R_1 \rightarrow R_1 - R_2$: $\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$
$R_2 \rightarrow R_2 - 2R_1$: $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$
$R_1 \rightarrow R_1 - R_2$: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}$

$\therefore A^{-1} = \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}$

Check: $|A| = 9 - 8 = 1$ and $AA^{-1} = I$.

Marking: Starting with $AA^{-1} = I$ and first transformation – ½ mark · Correct further row transformations – 1 mark · Final A^{-1} – ½ mark

Examiner tip: With row transformations, apply every operation to the right-hand matrix I as well, and use only rows (do not mix in column operations).

Q.5 Find the value of $\cos[\pi/4 + \tan^{-1}(-1/\sqrt{3})]$.

Answer (2 marks)

$\tan^{-1}(-1/\sqrt{3}) = -\pi/6$ (principal range of $\tan^{-1}$ is $(-\pi/2, \pi/2)$).

$\therefore$ Value $= \cos(\pi/4 - \pi/6) = \cos(\pi/4)\cos(\pi/6) + \sin(\pi/4)\sin(\pi/6)$

$= (1/\sqrt{2})(\sqrt{3}/2) + (1/\sqrt{2})(1/2) = \mathbf{(\sqrt{3} + 1)/(2\sqrt{2})}$ $[= (\sqrt{6} + \sqrt{2})/4]$

Marking: $\tan^{-1}(-1/\sqrt{3}) = -\pi/6$ – 1 mark · Expansion and final value – 1 mark

Examiner tip: $\tan^{-1}(-x) = -\tan^{-1}x$; taking $5\pi/6$ here is outside the principal range and gives the wrong answer.

Q.6 Find the separate equations of the lines represented by $10x^2 + 9xy - 9y^2 = 0$.

Answer (2 marks)

$10x^2 + 9xy - 9y^2 = 10x^2 + 15xy - 6xy - 9y^2$

$= 5x(2x + 3y) - 3y(2x + 3y)$

$= (5x - 3y)(2x + 3y)$

$\therefore$ The separate equations are **$5x - 3y = 0$ and $2x + 3y = 0$**.

Marking: Correct splitting of the middle term / factorisation – 1 mark · Both separate equations – 1 mark

Examiner tip: Multiply the factors back to check: $(5x - 3y)(2x + 3y)$ must give $+9xy$.

Q.7 Using vectors, show that the points A(2, −1, 3), B(4, 3, 1) and C(3, 1, 2) are collinear.

Answer (2 marks)

$\bar{AB} = (4 - 2)\hat{i} + (3 + 1)\hat{j} + (1 - 3)\hat{k} = 2\hat{i} + 4\hat{j} - 2\hat{k}$

$\bar{AC} = (3 - 2)\hat{i} + (1 + 1)\hat{j} + (2 - 3)\hat{k} = \hat{i} + 2\hat{j} - \hat{k}$

$\therefore \bar{AB} = 2\,\bar{AC}$, so $\bar{AB}$ and $\bar{AC}$ are parallel, and they have the common point A.

$\therefore$ **A, B, C are collinear**.

Marking: $\bar{AB}$ and $\bar{AC}$ – 1 mark · $\bar{AB} = 2\bar{AC}$ and conclusion (common point A) – 1 mark

Examiner tip: Parallel vectors alone are not enough: mention the common point A in the conclusion.

Q.8 The corner points of the feasible region of an L.P.P. are O(0, 0), A(6, 0), B(4, 3) and C(0, 5). Find the maximum value of $z = 4x + 5y$ and the point at which it occurs.

Answer (2 marks)

Corner point	$z = 4x + 5y$
O(0, 0)	0
A(6, 0)	24
B(4, 3)	31
C(0, 5)	25

$\therefore$ **Maximum $z = 31$ at $x = 4$, $y = 3$**, i.e. at B(4, 3).

Marking: Values of z at all four corners – 1 mark · Maximum value with the point – 1 mark

Examiner tip: The optimum of a bounded feasible region is at a corner; evaluate z at every corner, not just the largest-looking one.

Q.9 If $x = 4\sec\theta$ and $y = 3\tan\theta$, find dy/dx. Also find its value at $\theta = \pi/6$.

Answer (2 marks)

$dx/d\theta = 4\sec\theta\tan\theta$, $dy/d\theta = 3\sec^2\theta$

$dy/dx = (dy/d\theta)/(dx/d\theta) = 3\sec^2\theta / (4\sec\theta\tan\theta) = 3\sec\theta/(4\tan\theta) =$ **$3/(4\sin\theta)$**

At $\theta = \pi/6$: $dy/dx = 3/(4 \times 1/2) =$ **$3/2$**.

Marking: $dx/d\theta$ and $dy/d\theta$ – 1 mark · $dy/dx = 3/(4\sin\theta)$ and value $3/2$ – 1 mark

Examiner tip: $\sec\theta/\tan\theta = (1/\cos\theta)\cdot(\cos\theta/\sin\theta) = 1/\sin\theta$. Divide $dy/d\theta$ by $dx/d\theta$, not the other way round.

Q.10 Evaluate: $\int_0^{\log 2} e^x / [(e^x + 1)(e^x + 2)]\,dx$

Answer (2 marks)

Put $e^x = t \Rightarrow e^x\,dx = dt$; $x = 0 \Rightarrow t = 1$, $x = \log 2 \Rightarrow t = 2$.

$I = \int_1^2 dt/[(t + 1)(t + 2)] = \int_1^2 [1/(t + 1) - 1/(t + 2)]\,dt$

$= [\log(t + 1) - \log(t + 2)]_1^2 = [\log((t + 1)/(t + 2))]_1^2 = \log(3/4) - \log(2/3)$

$= \log[(3/4) \times (3/2)] \therefore$ **$I = \log(9/8)$**

Marking: Substitution with new limits and partial fractions – 1 mark · Evaluation $\log(9/8)$ – 1 mark

Examiner tip: Change the limits when you put $e^x = t$: $x = \log 2$ gives $t = e^{\log 2} = 2$.

Q.11 Complete the following activity to find the area of the region bounded by the curve $y = x^2 + 2$, the X-axis and the lines $x = 0$ and $x = 3$.

Activity: For $0 \le x \le 3$, $y = x^2 + 2 > 0$, so the region lies above the X-axis.

Required area = $\int_{\square}^3 (x^2 + 2)\,dx = [\square]_0^3 = 9 + \square = \square$ sq. units

Answer (2 marks)

Required area = $\int_{\mathbf{0}}^3 (x^2 + 2)\,dx = [\mathbf{x^3/3 + 2x}]_0^3 = 9 +$ **$6 =$ 15** sq. units

Marking: Lower limit 0 and integral $x^3/3 + 2x$ – 1 mark · Area 15 sq. units – 1 mark

Examiner tip: The curve lies above the X-axis for $0 \le x \le 3$, so the integral itself is the area; write 'sq. units' with the answer.

Q.12 Solve the differential equation $(x + 2)\,dy/dx = y + 1$, given that $y = 3$ when $x = 0$.

Answer (2 marks)

Separating the variables: $dy/(y + 1) = dx/(x + 2)$

Integrating: $\log|y + 1| = \log|x + 2| + \log c \Rightarrow y + 1 = c(x + 2)$

$x = 0$, $y = 3$: $4 = 2c \Rightarrow c = 2$

$\therefore y + 1 = 2(x + 2) \Rightarrow$ **$y = 2x + 3$**

Marking: Separating variables and general solution – 1 mark · Using $y(0) = 3$ and particular solution – 1 mark

Examiner tip: Write the constant as $\log c$; it makes combining the logarithms easy.

Q.13 A grocer has a tray of 6 eggs, of which 2 are cracked. Two eggs are picked at random without replacement. If X denotes the number of cracked eggs picked, find the probability distribution of X.

Answer (2 marks)

X can take the values 0, 1, 2. Total ways = ${}^6C_2 = 15$.

$P(X = 0) = {}^4C_2/15 = 6/15 = 2/5$

$P(X = 1) = ({}^2C_1 \times {}^4C_1)/15 = 8/15$

$P(X = 2) = {}^2C_2/15 = 1/15$

x	0	1	2
P(X = x)	2/5	8/15	1/15

Check: 6/15 + 8/15 + 1/15 = 1.

Marking: Possible values and total ways – ½ mark · Three probabilities – 1 mark · Table of the distribution – ½ mark

Examiner tip: Check that the probabilities add up to 1 before writing the table.

Q.14 A fair die is thrown 4 times. Getting a multiple of 3 is a success. Find the probability of exactly 2 successes.

Answer (2 marks)

Multiples of 3 on a die: 3, 6. p = 2/6 = 1/3, q = 2/3, n = 4. X ~ B(4, 1/3).

$P(X = 2) = {}^4C_2\,(1/3)^2\,(2/3)^2 = 6 \times (1/9) \times (4/9) = 24/81 =$ **8/27**

Marking: n, p, q and the binomial formula – 1 mark · P(X = 2) = 8/27 – 1 mark

Examiner tip: A multiple of 3 on a die means 3 or 6, so p = 1/3.

SECTION C Attempt any EIGHT of the following questions (Q.15 to Q.26, 3 marks each). **[24]**

Q.15 Using a truth table, show that $[(p \vee q) \wedge (p \to r) \wedge (q \to r)] \to r$ is a tautology.

Answer (3 marks)

p	q	r	p ∨ q	p → r	q → r	A: (p ∨ q) ∧ (p → r) ∧ (q → r)	A → r
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	T	T	T	T	T
T	F	F	T	F	T	F	T
F	T	T	T	T	T	T	T
F	T	F	T	T	F	F	T
F	F	T	F	T	T	F	T
F	F	F	F	T	T	F	T

Here A = (p ∨ q) ∧ (p → r) ∧ (q → r). All the entries in the last column A → r are T.

∴ **[(p ∨ q) ∧ (p → r) ∧ (q → r)] → r is a tautology**.

Marking: Columns p ∨ q, p → r and q → r – 1 mark · Column A = (p ∨ q) ∧ (p → r) ∧ (q → r) – 1 mark · Last column all T and conclusion – 1 mark

Examiner tip: A is T only in rows where r is also T, so A → r is never 'T → F'. Write the conclusion sentence.

Q.16 Find the co-ordinates of the foot of the perpendicular drawn from the point P(7, 5, 5) to the line (x − 1)/2 = (y − 2)/3 = (z − 3)/4. Also find the length of the perpendicular.

Answer (3 marks)

Any point on the line: M(2λ + 1, 3λ + 2, 4λ + 3).

PM direction ratios: 2λ − 6, 3λ − 3, 4λ − 2.

PM ⊥ line ⇒ 2(2λ − 6) + 3(3λ − 3) + 4(4λ − 2) = 0 ⇒ 29λ − 29 = 0 ⇒ λ = 1.

∴ Foot of the perpendicular **M(3, 5, 7)**.

$PM = \sqrt{[(7-3)^2 + (5-5)^2 + (5-7)^2]} = \sqrt{20} =$ **$2\sqrt{5}$ units**.

Marking: General point M and direction ratios of PM – 1 mark · Condition of perpendicularity, λ = 1, foot M(3, 5, 7) – 1 mark · Length $2\sqrt{5}$ units – 1 mark

Examiner tip: Use the direction ratios of the given line (2, 3, 4) in the perpendicularity condition.

Q.17 Two straight roads leave a village square C at an angle of 60°. Village B is 8 km from C along one road and village A is 15 km from C along the other. Taking $\triangle ABC$ with a = 8 km, b = 15 km and C = 60°, find (a) the distance AB = c (b) the area of $\triangle ABC$ (c) cos A.

Answer (3 marks)

(a) By the cosine rule: $c^2 = a^2 + b^2 - 2ab\cos C = 64 + 225 - 2(8)(15)(1/2) = 289 - 120 = 169$

$\therefore$ **c = AB = 13 km**

(b) $A(\triangle ABC) = (1/2)ab\sin C = (1/2)(8)(15)(\sqrt{3}/2) =$ **$30\sqrt{3}$ sq. km**

(c) $\cos A = (b^2 + c^2 - a^2)/(2bc) = (225 + 169 - 64)/(2 \times 15 \times 13) = 330/390 =$ **11/13**

Marking: c = 13 km by cosine rule – 1 mark · Area $30\sqrt{3}$ sq. km – 1 mark · cos A = 11/13 – 1 mark

Examiner tip: In the cosine rule the angle is the one between the two given sides (here C). Write units with each answer.

Q.18 Find the joint equation of the pair of lines passing through the point (1, −2) and parallel to the lines represented by $2x^2 - 5xy + 2y^2 = 0$.

Answer (3 marks)

$2x^2 - 5xy + 2y^2 = 2x^2 - 4xy - xy + 2y^2 = (2x - y)(x - 2y)$. The lines are 2x − y = 0 (slope 2) and x − 2y = 0 (slope 1/2).

Line through (1, −2) with slope 2: $y + 2 = 2(x - 1) \Rightarrow 2x - y - 4 = 0$.

Line through (1, −2) with slope 1/2: $2(y + 2) = x - 1 \Rightarrow x - 2y - 5 = 0$.

Joint equation: $(2x - y - 4)(x - 2y - 5) = 0$

$\therefore 2x^2 - 4xy - 10x - xy + 2y^2 + 5y - 4x + 8y + 20 = 0$

$\therefore$ **$2x^2 - 5xy + 2y^2 - 14x + 13y + 20 = 0$**

Marking: Separate lines and slopes 2, 1/2 – 1 mark · Two required lines – 1 mark · Joint equation – 1 mark

Examiner tip: Lines parallel to the given pair keep the second-degree terms $2x^2 - 5xy + 2y^2$; use this to check your expansion.

Q.19 A and B are the points with position vectors $\hat{i} + 4\hat{j} - 2\hat{k}$ and $5\hat{i} - 4\hat{j} + 6\hat{k}$. Find the position vectors of the points which divide the segment AB (a) internally (b) externally in the ratio 3 : 1.

Answer (3 marks)

Let $\bar{a} = \hat{i} + 4\hat{j} - 2\hat{k}$, $\bar{b} = 5\hat{i} - 4\hat{j} + 6\hat{k}$, m : n = 3 : 1.

(a) Internal division: $\bar{r} = (m\bar{b} + n\bar{a})/(m + n) = [3(5\hat{i} - 4\hat{j} + 6\hat{k}) + (\hat{i} + 4\hat{j} - 2\hat{k})]/4$

$= (16\hat{i} - 8\hat{j} + 16\hat{k})/4 =$ **$4\hat{i} - 2\hat{j} + 4\hat{k}$**

(b) External division: $\bar{r} = (m\bar{b} - n\bar{a})/(m - n) = [3(5\hat{i} - 4\hat{j} + 6\hat{k}) - (\hat{i} + 4\hat{j} - 2\hat{k})]/2$

$= (14\hat{i} - 16\hat{j} + 20\hat{k})/2 =$ **$7\hat{i} - 8\hat{j} + 10\hat{k}$**

Marking: Section formulae – 1 mark · Internal point $4\hat{i} - 2\hat{j} + 4\hat{k}$ – 1 mark · External point $7\hat{i} - 8\hat{j} + 10\hat{k}$ – 1 mark

Examiner tip: In m : n, m goes with the end point B. For external division the denominator is m − n.

Q.20 Find the vector and Cartesian equations of the plane passing through the point (1, −2, 3) and perpendicular to each of the planes 2x + y − z = 4 and x − y + 2z = 1.

Answer (3 marks)

Normals of the given planes: $\bar{n}_1 = 2\hat{i} + \hat{j} - \hat{k}$, $\bar{n}_2 = \hat{i} - \hat{j} + 2\hat{k}$.

The required plane is perpendicular to both planes, so its normal is

$$\bar{n} = \bar{n}_1 \times \bar{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -1 \\ 1 & -1 & 2 \end{vmatrix} = \hat{i}(2 - 1) - \hat{j}(4 + 1) + \hat{k}(-2 - 1) = \hat{i} - 5\hat{j} - 3\hat{k}$$

$\bar{a} = \hat{i} - 2\hat{j} + 3\hat{k}$, $\bar{a} \cdot \bar{n} = 1 + 10 - 9 = 2$

Vector equation: **$\bar{r} \cdot (\hat{i} - 5\hat{j} - 3\hat{k}) = 2$**

Cartesian equation: **$x - 5y - 3z = 2$**

Marking: Normal $\vec{n} = \vec{n}_1 \times \vec{n}_2$ – 1 mark · Vector equation – 1 mark · Cartesian equation – 1 mark

Examiner tip: A plane perpendicular to two planes has its normal perpendicular to both normals, so use their cross product.

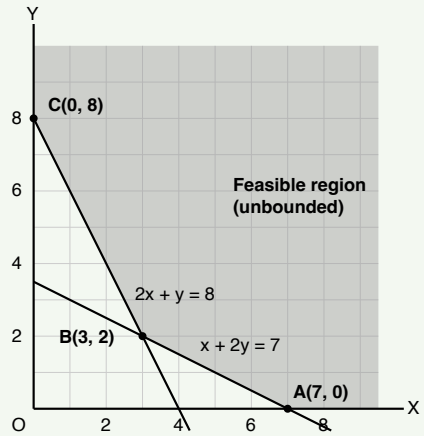
Q.21 Solve the following L.P.P. graphically:
Minimise $z = 3x + 5y$ subject to $2x + y \geq 8$, $x + 2y \geq 7$, $x \geq 0$, $y \geq 0$.

Answer (3 marks)

Draw $2x + y = 8$ (through (4, 0) and (0, 8)) and $x + 2y = 7$ (through (7, 0) and (0, 3.5)).
The origin does not satisfy $2x + y \geq 8$ or $x + 2y \geq 7$, so the feasible region lies on the side away from the origin. It is unbounded (shaded in the figure), with corners A(7, 0), B and C(0, 8).
B: $2x + y = 8$ and $x + 2y = 7 \Rightarrow y = 8 - 2x \Rightarrow x + 16 - 4x = 7 \Rightarrow x = 3, y = 2$. So B(3, 2).

Corner point	$z = 3x + 5y$
A(7, 0)	21
B(3, 2)	19
C(0, 8)	40

The smallest value is 19. The half-plane $3x + 5y < 19$ has no point in common with the feasible region, so 19 is the minimum.
$\therefore$ **z is minimum = 19 at x = 3, y = 2.**



Marking: Both lines and the feasible region – 1 mark · Corner points including B(3, 2) – 1 mark · Minimum z = 19 at (3, 2) – 1 mark

Examiner tip: For '≥' constraints the region is on the side away from the origin and is unbounded; mention why the smallest corner value is really the minimum.

Q.22 If $y = (2x + 1)^3 \sqrt{(x - 1)} / (x + 4)^2$, find dy/dx.

Answer (3 marks)

Taking logarithms: $\log y = 3 \log(2x + 1) + (1/2) \log(x - 1) - 2 \log(x + 4)$
Differentiating w.r.t. x:
$(1/y)\, dy/dx = 3 \times 2/(2x + 1) + (1/2) \times 1/(x - 1) - 2 \times 1/(x + 4)$
$\therefore$ **$dy/dx = [(2x + 1)^3 \sqrt{(x - 1)} / (x + 4)^2] [6/(2x + 1) + 1/(2(x - 1)) - 2/(x + 4)]$**

Marking: Taking logarithm and using log laws – 1 mark · Differentiating each term – 1 mark · Final dy/dx with y substituted – 1 mark

Examiner tip: Do not forget the factor 2 from the derivative of log(2x + 1), and multiply back by y at the end.

Q.23 Find the equations of the tangent and the normal to the curve $x^2 + xy + 2y^2 = 4$ at the point (1, 1).

Answer (3 marks)

Differentiating w.r.t. x: $2x + y + x\, dy/dx + 4y\, dy/dx = 0 \Rightarrow dy/dx = -(2x + y)/(x + 4y)$
At (1, 1): slope of tangent = $-3/5$.

Tangent: $y - 1 = (-3/5)(x - 1) \Rightarrow 5y - 5 = -3x + 3 \Rightarrow$ **$3x + 5y - 8 = 0$**

Slope of normal = 5/3.

Normal: $y - 1 = (5/3)(x - 1) \Rightarrow 3y - 3 = 5x - 5 \Rightarrow$ **$5x - 3y - 2 = 0$**

Marking: dy/dx by implicit differentiation, slope −3/5 – 1 mark · Equation of tangent – 1 mark · Equation of normal – 1 mark

Examiner tip: Check first that (1, 1) is on the curve: 1 + 1 + 2 = 4. Slope of normal = −1/(slope of tangent).

Q.24 Complete the following activity to evaluate $\int dx/(5 + 3\cos x)$.

Activity: Put $\tan(x/2) = t$, so $dx =$ ☐ and $\cos x = (1 - t^2)/(1 + t^2)$.

$5 + 3\cos x = [5(1 + t^2) + 3(1 - t^2)]/(1 + t^2) =$ ☐$/(1 + t^2)$

$I = \int 2\,dt/(8 + 2t^2) = \int dt/($☐$ + t^2) = (1/2)\tan^{-1}($☐$) + c$

$\therefore I =$ ☐ $+ c$

Answer (3 marks)

$dx =$ **$2\,dt/(1 + t^2)$**

$5 + 3\cos x =$ **$(8 + 2t^2)$**$/(1 + t^2)$

$I = \int dt/($**4**$ + t^2) = (1/2)\tan^{-1}($**$t/2$**$) + c$

$\therefore I =$ **$(1/2)\tan^{-1}[(1/2)\tan(x/2)]$** $+ c$

Marking: dx and denominator $8 + 2t^2$ – 1 mark · Reducing to $\int dt/(4 + t^2)$ – 1 mark · Final answer in x – 1 mark

Examiner tip: Return to x at the end: replace t by tan(x/2).

Q.25 Find the area of the region enclosed between the parabolas $y = x^2$ and $y = 8 - x^2$.

Answer (3 marks)

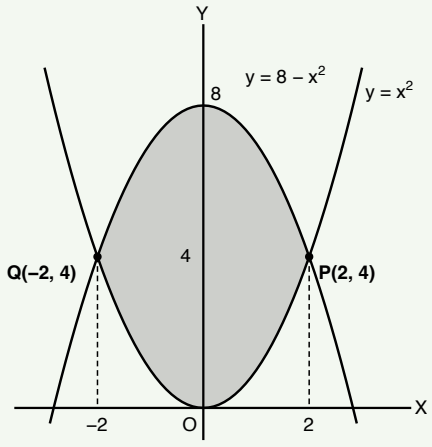
Points of intersection: $x^2 = 8 - x^2 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$. Points Q(−2, 4) and P(2, 4).

For $-2 \le x \le 2$ the curve $y = 8 - x^2$ lies above $y = x^2$ (see figure).

Required area $= \int_{-2}^{2} [(8 - x^2) - x^2]\,dx = \int_{-2}^{2} (8 - 2x^2)\,dx = 2\int_{0}^{2} (8 - 2x^2)\,dx$ (even function)

$= 2[8x - 2x^3/3]_0^2 = 2(16 - 16/3) = 2 \times 32/3$

$\therefore$ **Area = 64/3 sq. units**



Marking: Points of intersection (with rough sketch) – 1 mark · Correct integral set-up – 1 mark · Evaluation 64/3 sq. units – 1 mark

Examiner tip: $8 - 2x^2$ is an even function, so integrate from 0 to 2 and double. Write 'sq. units' with the answer.

Q.26 Solve the differential equation $x\,dy/dx = y + 2x\,e^{-y/x}$, given that $y = 0$ when $x = 1$.

Answer (3 marks)

$dy/dx = y/x + 2e^{-y/x}$ is homogeneous. Put $y = vx \Rightarrow dy/dx = v + x\,dv/dx$.

$v + x\,dv/dx = v + 2e^{-v} \Rightarrow x\,dv/dx = 2e^{-v}$

$\therefore e^v\,dv = 2\,dx/x$

Integrating: $e^v = 2\log x + c \Rightarrow e^{y/x} = 2\log x + c$

$x = 1, y = 0$: $e^0 = 2\log 1 + c \Rightarrow c = 1$

$\therefore$ **$e^{y/x} = 1 + 2\log x$** [i.e. $y = x\log(1 + 2\log x)$]

Marking: Substitution $y = vx$ and $x\,dv/dx = 2e^{-v}$ – 1 mark · Separating variables and integrating – 1 mark · Using $y(1) = 0$: particular solution $e^{y/x} = 1 + 2\log x$ – 1 mark

Examiner tip: Put $v = y/x$ back before using the condition; $\log 1 = 0$ and $e^0 = 1$.

SECTION D Attempt any FIVE of the following questions (Q.27 to Q.34, 4 marks each). **[20]**

Q.27 Solve the following equations by the method of inversion:
$x - y + z = 2,\ 2x + y - z = 1,\ x + 2y + z = 11$

Answer (4 marks)

$AX = B$ with $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 2 \\ 1 \\ 11 \end{bmatrix}$

$|A| = 1(1 + 2) + 1(2 + 1) + 1(4 - 1) = 3 + 3 + 3 = 9 \neq 0$, so A^{-1} exists.

Cofactors: $A_{11} = 3, A_{12} = -3, A_{13} = 3; A_{21} = 3, A_{22} = 0, A_{23} = -3; A_{31} = 0, A_{32} = 3, A_{33} = 3$

$\text{adj } A = \begin{bmatrix} 3 & 3 & 0 \\ -3 & 0 & 3 \\ 3 & -3 & 3 \end{bmatrix}$

$A^{-1} = (1/9)\,\text{adj } A$

$X = A^{-1}B = (1/9)\begin{bmatrix} 3 & 3 & 0 \\ -3 & 0 & 3 \\ 3 & -3 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 11 \end{bmatrix} = (1/9)\begin{bmatrix} 9 \\ 27 \\ 36 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$

$\therefore$ **$x = 1, y = 3, z = 4$**

Check: $2 + 3 - 4 = 1$ ✓; $1 + 6 + 4 = 11$ ✓

Marking: $|A| = 9$ – ½ mark · Cofactors and adj A – 1½ marks · $X = A^{-1}B$ multiplication – 1 mark · $x = 1, y = 3, z = 4$ – 1 mark

Examiner tip: adj A is the transpose of the cofactor matrix. Check the answer in all three equations.

Q.28 Find the general solution of $\sqrt{3}\sin\theta - \cos\theta = \sqrt{2}$. Hence write its principal solutions.

Answer (4 marks)

Dividing by 2: $(\sqrt{3}/2)\sin\theta - (1/2)\cos\theta = 1/\sqrt{2}$

$\therefore \sin\theta\cos(\pi/6) - \cos\theta\sin(\pi/6) = 1/\sqrt{2} \Rightarrow \sin(\theta - \pi/6) = \sin(\pi/4)$

If $\sin\alpha = \sin\beta$, then $\alpha = n\pi + (-1)^n\beta$, $n \in Z$.

$\therefore \theta - \pi/6 = n\pi + (-1)^n\,\pi/4$

$\therefore$ General solution: **$\theta = n\pi + (-1)^n\,\pi/4 + \pi/6,\ n \in Z$**

Principal solutions ($0 \le \theta < 2\pi$): $n = 0 \Rightarrow \theta = \pi/4 + \pi/6 = 5\pi/12$; $n = 1 \Rightarrow \theta = \pi - \pi/4 + \pi/6 = 11\pi/12$ ($n = 2$ gives $29\pi/12 > 2\pi$).

$\therefore$ **Principal solutions: $5\pi/12, 11\pi/12$**

Marking: Dividing by $\sqrt{(3 + 1)} = 2$ and writing $\sin(\theta - \pi/6)$ – 1 mark · $\sin(\theta - \pi/6) = \sin(\pi/4)$ – 1 mark · General solution with $n \in Z$ – 1 mark · Principal solutions $5\pi/12, 11\pi/12$ – 1 mark

Examiner tip: Divide by $\sqrt{(a^2 + b^2)} = 2$ first. Check a principal solution: $\sqrt{3}\sin(5\pi/12) - \cos(5\pi/12) = 2\sin(\pi/4) = \sqrt{2}$.

Q.29 A(1, 0, 2), B(3, 1, 1), C(2, 3, 4) and D(4, 2, 5) are four points in space. Find (a) the area of △ABC (b) the volume of the tetrahedron ABCD.

Answer (4 marks)

$\bar{AB} = 2\hat{i} + \hat{j} - \hat{k}$, $\bar{AC} = \hat{i} + 3\hat{j} + 2\hat{k}$, $\bar{AD} = 3\hat{i} + 2\hat{j} + 3\hat{k}$

(a) $\bar{AB} \times \bar{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -1 \\ 1 & 3 & 2 \end{vmatrix} = \hat{i}(2+3) - \hat{j}(4+1) + \hat{k}(6-1) = 5\hat{i} - 5\hat{j} + 5\hat{k}$

$|\bar{AB} \times \bar{AC}| = \sqrt{75} = 5\sqrt{3}$

$A(\triangle ABC) = (1/2)|\bar{AB} \times \bar{AC}| =$ **$(5\sqrt{3})/2$ sq. units**

(b) $[\bar{AB}\ \bar{AC}\ \bar{AD}] = \begin{vmatrix} 2 & 1 & -1 \\ 1 & 3 & 2 \\ 3 & 2 & 3 \end{vmatrix} = 2(9-4) - 1(3-6) - 1(2-9) = 10 + 3 + 7 = 20$

Volume of tetrahedron $= (1/6)|[\bar{AB}\ \bar{AC}\ \bar{AD}]| = 20/6 =$ **10/3 cubic units**

Marking: Vectors $\bar{AB}$, $\bar{AC}$, $\bar{AD}$ – ½ mark · Cross product and area $(5\sqrt{3})/2$ – 1½ marks · Scalar triple product = 20 – 1 mark · Volume 10/3 cubic units – 1 mark

Examiner tip: Area of a triangle is half the magnitude of the cross product; volume of a tetrahedron is one-sixth of the scalar triple product.

Q.30 A farmer wants to fence a rectangular cattle pen of area 800 m^2 next to the straight stone wall of his farmhouse. No fencing is needed along the wall. Find the dimensions of the pen for which the length of fencing is least. Also find the cost of this fencing at ₹ 150 per metre.

Answer (4 marks)

Let the side perpendicular to the wall be y m and the side parallel to the wall be x m. Then $xy = 800 \Rightarrow x = 800/y$.

Fencing $L = x + 2y = 800/y + 2y$, $y > 0$

$dL/dy = -800/y^2 + 2 = 0 \Rightarrow y^2 = 400 \Rightarrow y = 20$ $(y > 0)$

$d^2L/dy^2 = 1600/y^3$; at $y = 20$ it is $1600/8000 = 1/5 > 0$, so L is minimum.

$x = 800/20 = 40$, $L_{min} = 40 + 40 = 80$ m.

∴ **Pen: 40 m along the wall and 20 m perpendicular to it; least fencing 80 m**

Cost $= 80 \times 150 =$ **₹ 12,000**

Marking: Expression $L = 800/y + 2y$ – 1 mark · $dL/dy = 0 \Rightarrow y = 20$ – 1 mark · Second derivative test (minimum) – 1 mark · Dimensions, 80 m and cost ₹ 12,000 – 1 mark

Examiner tip: Only three sides are fenced, so $L = x + 2y$, not $2x + 2y$. Show the second derivative test for the minimum.

Q.31 Complete the following activity to evaluate $I = \int_0^{\pi/4} x \tan^2 x\, dx$.

Activity: $\tan^2 x =$ ☐ $- 1$, so $I = \int_0^{\pi/4} x \sec^2 x\, dx - \int_0^{\pi/4} x\, dx$

Integrating by parts: $\int x \sec^2 x\, dx = x \tan x - \int$ ☐ $dx = x \tan x + \log(\cos x) + c$

$\therefore \int_0^{\pi/4} x \sec^2 x\, dx = [x \tan x + \log(\cos x)]_0^{\pi/4} = \pi/4 +$ ☐

$\int_0^{\pi/4} x\, dx = [x^2/2]_0^{\pi/4} =$ ☐

$\therefore I =$ ☐

Answer (4 marks)

$\tan^2 x =$ **$\sec^2 x$** $- 1$, so $I = \int_0^{\pi/4} x \sec^2 x\, dx - \int_0^{\pi/4} x\, dx$

By parts ($u = x$, $v = \sec^2 x$): $\int x \sec^2 x\, dx = x \tan x - \int$ **$\tan x$** $dx = x \tan x - \log(\sec x) = x \tan x + \log(\cos x) + c$

$[x \tan x + \log(\cos x)]_0^{\pi/4} = (\pi/4)(1) + \log(1/\sqrt{2}) - (0 + \log 1) = \pi/4 +$ **$-(1/2) \log 2$)**

$\int_0^{\pi/4} x\,dx = (1/2)(\pi/4)^2 = \boldsymbol{\pi^2/32}$

$\therefore$ **$I = \pi/4 - (1/2)\log 2 - \pi^2/32$**

Marking: $\tan^2 x = \sec^2 x - 1$ and splitting the integral – 1 mark · Integration by parts: $x\tan x - \int \tan x\,dx = x\tan x + \log(\cos x)$ – 1 mark · Value $\pi/4 - (1/2)\log 2$ of the first integral and $\pi^2/32$ of the second – 1 mark · $I = \pi/4 - (1/2)\log 2 - \pi^2/32$ – 1 mark

Examiner tip: Write $\tan^2 x$ as $\sec^2 x - 1$ first; $\int x\tan^2 x\,dx$ cannot be done by parts directly. At the end, $\log(\cos \pi/4) = \log(1/\sqrt{2}) = -(1/2)\log 2$ — the sign is where most marks are lost.

Q.32 Evaluate: $\int (3x + 2)/(x^2 + 4x + 13)\,dx$

Answer (4 marks)

$d/dx\,(x^2 + 4x + 13) = 2x + 4$. Write $3x + 2 = (3/2)(2x + 4) - 4$.

$I = (3/2)\int (2x + 4)/(x^2 + 4x + 13)\,dx - 4\int dx/(x^2 + 4x + 13)$

First integral: $(3/2)\log|x^2 + 4x + 13|$

$x^2 + 4x + 13 = (x + 2)^2 + 3^2$, so $\int dx/[(x + 2)^2 + 3^2] = (1/3)\tan^{-1}[(x + 2)/3]$

$\therefore$ **$I = (3/2)\log|x^2 + 4x + 13| - (4/3)\tan^{-1}[(x + 2)/3] + c$**

Marking: Writing $3x + 2 = (3/2)(2x + 4) - 4$ – 1 mark · First integral (log term) – 1 mark · Completing the square and $\tan^{-1}$ term – 1 mark · Final answer with c – 1 mark

Examiner tip: Check the split: $(3/2)(2x + 4) - 4 = 3x + 6 - 4 = 3x + 2$.

Q.33 A cup of tea at 90 °C is kept in a room whose temperature is 26 °C. It cools to 58 °C in 10 minutes. Using Newton's law of cooling, find (a) the temperature of the tea after 20 minutes (b) the time it takes to cool to 34 °C.

Answer (4 marks)

Let θ be the temperature at time t minutes. By Newton's law of cooling $d\theta/dt = -k(\theta - 26)$, $k > 0$.

$\int d\theta/(\theta - 26) = -k\int dt \Rightarrow \log(\theta - 26) = -kt + c$

$t = 0, \theta = 90 \Rightarrow c = \log 64 \Rightarrow \theta - 26 = 64\,e^{-kt}$ ...(1)

$t = 10, \theta = 58 \Rightarrow 32 = 64\,e^{-10k} \Rightarrow e^{-10k} = 1/2 \Rightarrow e^{-kt} = (1/2)^{t/10}$

$\therefore \theta = 26 + 64\,(1/2)^{t/10}$

(a) $t = 20$: $\theta = 26 + 64 \times (1/4) =$ **42 °C**

(b) $\theta = 34$: $8 = 64\,(1/2)^{t/10} \Rightarrow (1/2)^{t/10} = 1/8 = (1/2)^3 \Rightarrow t/10 = 3 \Rightarrow$ **t = 30 minutes**

Marking: Forming $d\theta/dt = -k(\theta - 26)$ – 1 mark · Solving and using $\theta(0) = 90$ – 1 mark · $e^{-10k} = 1/2$ and θ after 20 min = 42 °C – 1 mark · Time 30 minutes – 1 mark

Examiner tip: The rate depends on $\theta - 26$ (the excess over room temperature), not on θ. Keep $e^{-10k} = 1/2$; no log table is needed.

Q.34 The p.d.f. of a continuous random variable X is $f(x) = k(4x - x^2)$ for $0 \le x \le 4$, and $f(x) = 0$ otherwise. Find (a) k (b) the c.d.f. $F(x)$ for $0 \le x \le 4$ (c) $P(1 < X < 3)$.

Answer (4 marks)

(a) $\int_0^4 k(4x - x^2)\,dx = 1 \Rightarrow k[2x^2 - x^3/3]_0^4 = k(32 - 64/3) = (32/3)k = 1 \Rightarrow$ **$k = 3/32$**

(b) For $0 \le x \le 4$: $F(x) = \int_0^x (3/32)(4t - t^2)\,dt = (3/32)(2x^2 - x^3/3)$

$\therefore$ **$F(x) = (3/16)x^2 - x^3/32$** ($F(x) = 0$ for $x < 0$ and 1 for $x > 4$)

(c) $P(1 < X < 3) = F(3) - F(1) = (27/16 - 27/32) - (3/16 - 1/32) = 27/32 - 5/32 = 22/32 =$ **11/16**

Marking: Total probability = 1 and $k = 3/32$ – 1 mark · c.d.f. $F(x)$ – 1½ marks · $P(1 < X < 3) = 11/16$ – 1½ marks

Examiner tip: Check $F(4) = 3 - 2 = 1$. For a continuous variable $P(1 < X < 3) = F(3) - F(1)$.

All the best for your exam!