Mathematics and Statistics (Arts and Science)

HSC Board Exam February–March 2027 – Practice Paper

Model Answer Paper

Time: 3 hours **Max. Marks: 80**

Model answers show one correct method; any other correct method gets full marks.

SECTION A – Q.1 Select and write the correct answer for the following multiple choice type of questions. **[16]**

(i) The negation of the statement pattern $(p \wedge \sim q) \rightarrow r$ is .....

(a) $(p \wedge \sim q) \wedge \sim r$ (b) $(\sim p \vee q) \wedge \sim r$

(c) $(p \wedge \sim q) \vee r$ (d) $\sim p \vee q \vee r$

Answer (2 marks)

(a) $(p \wedge \sim q) \wedge \sim r$

$\sim(A \rightarrow B) \equiv A \wedge \sim B$. Here $A = p \wedge \sim q$ and $B = r$, so $\sim[(p \wedge \sim q) \rightarrow r] \equiv (p \wedge \sim q) \wedge \sim r$.

Marking: Correct alternative with its alphabet – 2 marks

Examiner tip: Option (b) negates the antecedent as well. In $\sim(A \rightarrow B)$ only the consequent is negated; A stays as it is.

(ii) If $A = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$ and $A \cdot (\text{adj } A) = kI$, where I is the identity matrix of order 2, then k = ...

(a) 14 (b) 26

(c) −14 (d) 196

Answer (2 marks)

(a) 14

$A \cdot (\text{adj } A) = |A| \cdot I$ and $|A| = \begin{vmatrix} 4 & 3 \\ 2 & 5 \end{vmatrix} = 20 - 6 = 14$. So $k = 14$.

Marking: Correct alternative with its alphabet – 2 marks

(iii) The principal value of $\sin^{-1}(-\sqrt{3}/2) + \cot^{-1}(-1/\sqrt{3})$ is .....

(a) $-2\pi/3$ (b) $\pi/3$

(c) $-\pi/3$ (d) π

Answer (2 marks)

(b) $\pi/3$

$\sin^{-1}(-\sqrt{3}/2) = -\pi/3$ (principal range of $\sin^{-1}$ is $[-\pi/2, \pi/2]$).

$\cot^{-1}(-1/\sqrt{3}) = \pi - \cot^{-1}(1/\sqrt{3}) = \pi - \pi/3 = 2\pi/3$ (principal range of $\cot^{-1}$ is $(0, \pi)$).

$\therefore -\pi/3 + 2\pi/3 = \pi/3$.

Marking: Correct alternative with its alphabet – 2 marks

Examiner tip: $\cot^{-1}(-1/\sqrt{3})$ is not $-\pi/3$: the principal value of $\cot^{-1}$ always lies in $(0, \pi)$.

(iv) The acute angle between the lines represented by $3x^2 - 7xy + 2y^2 = 0$ is .....

(a) 30° (b) 45°

(c) 60° (d) 90°

Answer (2 marks)

(b) 45°

$a = 3, 2h = -7 (h = -7/2), b = 2.$

$\tan\theta = |2\sqrt{(h^2 - ab)} / (a + b)| = |2\sqrt{(49/4 - 6)} / 5| = |2 \times (5/2) / 5| = 1 \Rightarrow \theta = 45°.$

Marking: Correct alternative with its alphabet – 2 marks

(v) If $|\bar{a}| = 3$, $|\bar{b}| = 4$ and $\bar{a} \cdot \bar{b} = 6$, then $|\bar{a} \times \bar{b}| =$ .....

(a) 6 (b) $6\sqrt{3}$

(c) 12 (d) $2\sqrt{3}$

Answer (2 marks)

(b) $6\sqrt{3}$

$|\bar{a} \times \bar{b}|^2 = |\bar{a}|^2|\bar{b}|^2 - (\bar{a} \cdot \bar{b})^2 = 9 \times 16 - 36 = 108.$

$\therefore |\bar{a} \times \bar{b}| = \sqrt{108} = 6\sqrt{3}.$

Marking: Correct alternative with its alphabet – 2 marks

(vi) If $y = \log[e^x/(1 + e^x)]$, then $dy/dx =$ .....

(a) $e^x/(1 + e^x)$ (b) $1/(1 + e^x)$

(c) 1 (d) $-1/(1 + e^x)$

Answer (2 marks)

(b) $1/(1 + e^x)$

$y = \log e^x - \log(1 + e^x) = x - \log(1 + e^x)$

$\therefore dy/dx = 1 - e^x/(1 + e^x) = 1/(1 + e^x).$

Marking: Correct alternative with its alphabet – 2 marks

(vii) $\int_{-2}^{2} (x^3 \cos x + 4)\, dx =$ .....

(a) 0 (b) 8

(c) 16 (d) 4

Answer (2 marks)

(c) 16

$x^3 \cos x$ is an odd function, so $\int_{-2}^{2} x^3 \cos x\, dx = 0.$

$\int_{-2}^{2} 4\, dx = 4[x]_{-2}^{2} = 4 \times 4 = 16.$

Marking: Correct alternative with its alphabet – 2 marks

(viii) For a binomial distribution $X \sim B(n, p)$, the mean is 9 and the variance is 6.3. Then $n =$ .....

(a) 15 (b) 20

(c) 25 (d) 30

Answer (2 marks)

(d) 30

$np = 9$ and $npq = 6.3 \Rightarrow q = 6.3/9 = 0.7, p = 0.3.$

$n = 9/0.3 = 30.$

Marking: Correct alternative with its alphabet – 2 marks

Q.2 Answer the following questions. **[4]**

(i) Write the truth value of the statement: "If 5 is an even number, then 9 is a prime number."

Answer (1 mark)
The antecedent "5 is an even number" is false (F) and the consequent "9 is a prime number" is false (F).
$F \rightarrow F$ is true.
$\therefore$ Truth value: **T**.

(ii) Find the distance of the point (1, 2, −3) from the plane $2x + y - 2z + 4 = 0$.

Answer (1 mark)
$d = |2(1) + 2 - 2(-3) + 4| \,/\, \sqrt{(4 + 1 + 4)} = |14| \,/\, 3$
$\therefore d =$ **14/3 units**.

(iii) Evaluate: $\int (\sin x + \cos x)^2 \, dx$

Answer (1 mark)
$\int (\sin^2 x + \cos^2 x + 2 \sin x \cos x) \, dx = \int (1 + \sin 2x) \, dx =$ **$x - (\cos 2x)/2 + c$**.

(iv) The p.d.f. of a continuous random variable X is $f(x) = kx^2$ for $0 \le x \le 3$, and $f(x) = 0$ otherwise. Find k.

Answer (1 mark)
$\int_0^3 kx^2 \, dx = 1 \Rightarrow k[x^3/3]_0^3 = 9k = 1$
$\therefore$ **$k = 1/9$**.

SECTION B Attempt any EIGHT of the following questions (Q.3 to Q.14, 2 marks each). **[16]**

Q.3 Without using a truth table, show that $\sim[p \vee (\sim p \wedge q)] \equiv \sim p \wedge \sim q$.

Answer (2 marks)
$p \vee (\sim p \wedge q) \equiv (p \vee \sim p) \wedge (p \vee q)$ ..... (Distributive law)
$\equiv T \wedge (p \vee q)$ ..... (Complement law)
$\equiv p \vee q$ ..... (Identity law)
$\therefore$ L.H.S. $= \sim[p \vee (\sim p \wedge q)] \equiv \sim(p \vee q)$
$\equiv \sim p \wedge \sim q$ ..... (De Morgan's law)
= R.H.S.

Marking: Distributive law step – ½ mark · Complement and identity laws to get $p \vee q$ – 1 mark · De Morgan's law to reach $\sim p \wedge \sim q$ – ½ mark

Examiner tip: Name the law used at every step. Examiners give the step marks only when the law is written next to the step.

Q.4 Find the inverse of $A = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$ by the adjoint method.

Answer (2 marks)
$|A| = \begin{vmatrix} 3 & -2 \\ 4 & -1 \end{vmatrix} = -3 + 8 = 5 \neq 0$, so A^{-1} exists.
Cofactors: $A_{11} = -1$, $A_{12} = -4$, $A_{21} = 2$, $A_{22} = 3$.
$\text{adj } A = \begin{bmatrix} -1 & 2 \\ -4 & 3 \end{bmatrix}$
$A^{-1} = (1/|A|) \text{ adj } A = (1/5) \begin{bmatrix} -1 & 2 \\ -4 & 3 \end{bmatrix}$

Marking: $|A| = 5$ – ½ mark · Correct adj A – 1 mark · $A^{-1} = (1/5)$ adj A – ½ mark

Examiner tip: adj A is the transpose of the cofactor matrix. A common slip is to forget the transpose or the sign of A_{12}.

Q.5 Find the general solution of $2\sin^2\theta + 5\cos\theta - 4 = 0$.

Answer (2 marks)

$2(1 - \cos^2\theta) + 5\cos\theta - 4 = 0$

$\therefore 2\cos^2\theta - 5\cos\theta + 2 = 0$

$\therefore (2\cos\theta - 1)(\cos\theta - 2) = 0$

$\cos\theta = 2$ is impossible ($|\cos\theta| \le 1$), so $\cos\theta = 1/2 = \cos(\pi/3)$.

General solution: $\boldsymbol{\theta = 2n\pi \pm \pi/3,\ n \in Z}$.

Marking: Quadratic in $\cos\theta$ and factorisation – 1 mark · Rejecting $\cos\theta = 2$ and correct general solution – 1 mark

Examiner tip: Write the reason for rejecting $\cos\theta = 2$. Also state $n \in Z$, which is often left out.

Q.6 Find the joint equation of the pair of lines through the origin which are perpendicular to the lines $5x - 2y = 0$ and $x + 3y = 0$.

Answer (2 marks)

Slope of $5x - 2y = 0$ is $5/2$, so the perpendicular line through the origin has slope $-2/5$: $2x + 5y = 0$.

Slope of $x + 3y = 0$ is $-1/3$, so the perpendicular line through the origin has slope 3: $3x - y = 0$.

Joint equation: $(2x + 5y)(3x - y) = 0$

$\therefore 6x^2 - 2xy + 15xy - 5y^2 = 0$

$\therefore \boldsymbol{6x^2 + 13xy - 5y^2 = 0}$.

Marking: Both perpendicular lines – 1 mark · Correct joint equation – 1 mark

Examiner tip: The slope of a line perpendicular to one with slope m is $-1/m$. Multiply the two equations out fully; the xy term is where most slips happen.

Q.7 Find a unit vector perpendicular to both $\bar{a} = \hat{i} + 2\hat{j} - \hat{k}$ and $\bar{b} = 2\hat{i} - \hat{j} + 3\hat{k}$.

Answer (2 marks)

$$\bar{a} \times \bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -1 \\ 2 & -1 & 3 \end{vmatrix} = \hat{i}(6 - 1) - \hat{j}(3 + 2) + \hat{k}(-1 - 4) = 5\hat{i} - 5\hat{j} - 5\hat{k}$$

$|\bar{a} \times \bar{b}| = \sqrt{(25 + 25 + 25)} = 5\sqrt{3}$

Required unit vector $= \pm(5\hat{i} - 5\hat{j} - 5\hat{k})/(5\sqrt{3}) = \boldsymbol{\pm(1/\sqrt{3})(\hat{i} - \hat{j} - \hat{k})}$.

Marking: Cross product – 1 mark · Magnitude and unit vector – 1 mark

Examiner tip: Both signs (±) give a valid answer; writing either is accepted, but do not forget to divide by the magnitude.

Q.8 Find the angle between the planes $2x - y + z = 6$ and $x + y + 2z = 7$.

Answer (2 marks)

Normals: $\bar{n}_1 = (2, -1, 1)$, $\bar{n}_2 = (1, 1, 2)$.

$\cos\theta = |\bar{n}_1 \cdot \bar{n}_2| \,/\, (|\bar{n}_1||\bar{n}_2|) = |2 - 1 + 2| \,/\, (\sqrt{6} \cdot \sqrt{6}) = 3/6 = 1/2$

$\therefore \boldsymbol{\theta = 60^\circ\ (\pi/3)}$.

Marking: Normal vectors and formula – 1 mark · Correct angle – 1 mark

Q.9 If $y = \tan^{-1}[4x / (1 - 4x^2)]$, where $|x| < 1/2$, find dy/dx.

Answer (2 marks)

Put $2x = \tan\theta$, so $\theta = \tan^{-1}(2x)$ and $-\pi/4 < \theta < \pi/4$.

$y = \tan^{-1}[2\tan\theta / (1 - \tan^2\theta)] = \tan^{-1}(\tan 2\theta) = 2\theta$ (since $-\pi/2 < 2\theta < \pi/2$)

$\therefore y = 2\tan^{-1}(2x)$

$\therefore dy/dx = 2 \times 2/(1 + 4x^2) =$ **$4/(1 + 4x^2)$.**

Marking: Substitution $2x = \tan\theta$ and simplification to $2\tan^{-1}(2x)$ – 1 mark · Correct derivative (chain rule factor 2) – 1 mark

Examiner tip: Mention why $\tan^{-1}(\tan 2\theta) = 2\theta$ (the interval). Direct differentiation by the chain rule is also accepted.

Q.10 Complete the following activity to find the approximate value of $\sqrt{81.36}$.

Activity: Let $f(x) = \sqrt{x}$, $a = 81$, $h =$ ☐

$f'(x) =$ ☐

$f(a) = \sqrt{81} = 9$ and $f'(a) =$ ☐

$f(a + h) \approx f(a) + h \cdot f'(a)$

$\therefore \sqrt{81.36} \approx 9 + 0.36 \times$ ☐ $=$ ☐

Answer (2 marks)

$h =$ **0.36**

$f'(x) =$ **$1/(2\sqrt{x})$**

$f'(a) = 1/(2 \times 9) =$ **1/18**

$\sqrt{81.36} \approx 9 + 0.36 \times$ **1/18** $= 9 + 0.02 =$ **9.02**

Marking: h and $f'(x)$ – ½ mark · $f'(81) = 1/18$ – ½ mark · Final value 9.02 – 1 mark

Q.11 Evaluate: $\int x \cos 4x\, dx$

Answer (2 marks)

By integration by parts ($u = x$, $v = \cos 4x$):

$\int x \cos 4x\, dx = x(\sin 4x / 4) - \int 1 \cdot (\sin 4x / 4)\, dx$

$= (x \sin 4x)/4 - (1/4)\int \sin 4x\, dx$

$=$ **$(x \sin 4x)/4 + (\cos 4x)/16 + c$**

Marking: Correct use of the by-parts formula – 1 mark · Final answer with constant c – 1 mark

Examiner tip: Take u = x (the algebraic function) as the first function (LIATE rule). Do not forget +c.

Q.12 Evaluate: $\int_0^{\pi/4} \sec^2 x / (3 + \tan x)\, dx$

Answer (2 marks)

Put $3 + \tan x = t \Rightarrow \sec^2 x\, dx = dt$.

When $x = 0$, $t = 3$; when $x = \pi/4$, $t = 4$.

$I = \int_3^4 dt/t = [\log t]_3^4 = \log 4 - \log 3$

$\therefore$ **$I = \log(4/3)$**

Marking: Substitution with changed limits – 1 mark · Evaluation – 1 mark

Examiner tip: When you substitute, change the limits too (or go back to x before putting the limits).

Q.13 Obtain the differential equation by eliminating the arbitrary constants A and B from $y = Ae^{3x} + Be^{-2x}$.

Answer (2 marks)

$y = Ae^{3x} + Be^{-2x}$ ...(1)

$dy/dx = 3Ae^{3x} - 2Be^{-2x}$ ...(2)

$d^2y/dx^2 = 9Ae^{3x} + 4Be^{-2x}$ ...(3)

(3) − (2): $d^2y/dx^2 - dy/dx = 6Ae^{3x} + 6Be^{-2x} = 6y$ ...by (1)

$\therefore$ **$d^2y/dx^2 - dy/dx - 6y = 0$**

Marking: First and second derivatives – 1 mark · Eliminating A, B to get the equation – 1 mark

Examiner tip: Two arbitrary constants means differentiate twice; the final equation must contain no A or B.

Q.14 The probability distribution of a discrete random variable X is:

x	1	2	3
P(X = x)	0.2	0.5	0.3

Find E(X) and Var(X).

Answer (2 marks)

$E(X) = \Sigma\, x \cdot P(x) = 1(0.2) + 2(0.5) + 3(0.3) = 0.2 + 1.0 + 0.9 = \mathbf{2.1}$

$E(X^2) = 1(0.2) + 4(0.5) + 9(0.3) = 0.2 + 2.0 + 2.7 = 4.9$

$Var(X) = E(X^2) - [E(X)]^2 = 4.9 - 4.41 = \mathbf{0.49}$

Marking: E(X) = 2.1 – 1 mark · Var(X) = 0.49 – 1 mark

Examiner tip: Var(X) is $E(X^2)$ minus the square of E(X), not $E(X^2)$ alone.

SECTION C Attempt any EIGHT of the following questions (Q.15 to Q.26, 3 marks each). **[24]**

Q.15 Using a truth table, prove that $\sim p \rightarrow (q \rightarrow r) \equiv (q \wedge \sim r) \rightarrow p$.

Answer (3 marks)

p	**q**	**r**	**~p**	**q → r**	**~p → (q → r)**	**~r**	**q ∧ ~r**	**(q ∧ ~r) → p**
T	T	T	F	T	T	F	F	T
T	T	F	F	F	T	T	T	T
T	F	T	F	T	T	F	F	T
T	F	F	F	T	T	T	F	T
F	T	T	T	T	T	F	F	T
F	T	F	T	F	F	T	T	F
F	F	T	T	T	T	F	F	T
F	F	F	T	T	T	T	F	T

The columns of $\sim p \rightarrow (q \rightarrow r)$ and $(q \wedge \sim r) \rightarrow p$ are identical.

$\therefore$ **$\sim p \rightarrow (q \rightarrow r) \equiv (q \wedge \sim r) \rightarrow p$.**

Marking: Columns ~p, q → r, ~r and q ∧ ~r – 1 mark · Correct final two columns (all 8 rows) – 1½ marks · Conclusion – ½ mark

Examiner tip: With three statements the table needs $2^3 = 8$ rows. An implication is false only when the antecedent is T and the consequent is F. Write the conclusion sentence; it carries a mark.

Q.16 Complete the following activity to solve the equations $2x + 3y = 13$ and $3x - 2y = 0$ by the method of inversion.

Activity: The equations can be written as $AX = B$, where $A = \begin{bmatrix} 2 & 3 \\ 3 & -2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 13 \\ 0 \end{bmatrix}$

$|A| = \square \neq 0$, so A^{-1} exists.

adj $A = \begin{bmatrix} \square & -3 \\ -3 & \square \end{bmatrix}$

$A^{-1} = (1/|A|)$ adj $A = (-1/13)$ adj A

(adj $A)B = \begin{bmatrix} \square \\ \square \end{bmatrix}$

$X = A^{-1}B = (-1/13)$(adj $A)B$

$\therefore x = \square$, $y = \square$

Answer (3 marks)

$|A| = -4 - 9 =$ **-13**

adj $A = \begin{bmatrix} \mathbf{-2} & -3 \\ -3 & \mathbf{2} \end{bmatrix}$

(adj $A)B$: first entry $(-2)(13) + (-3)(0) = -26$; second entry $(-3)(13) + (2)(0) = -39$

(adj $A)B = \begin{bmatrix} \mathbf{-26} \\ \mathbf{-39} \end{bmatrix}$

$X = \begin{bmatrix} x \\ y \end{bmatrix} = (-1/13)\begin{bmatrix} -26 \\ -39 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$\therefore$ **$x = 2, y = 3$**

Check: $2(2) + 3(3) = 13$ ✓, $3(2) - 2(3) = 0$ ✓

Marking: $|A| = -13$ – ½ mark · adj A – 1 mark · Product $A^{-1}B$ – 1 mark · $x = 2, y = 3$ – ½ mark

Examiner tip: Multiply row by column carefully: the first entry is $(-2)(13) + (-3)(0) = -26$.

Q.17 Show that $\sin^{-1}(5/13) + \tan^{-1}(7/17) = \pi/4$.

Answer (3 marks)

Let $\sin^{-1}(5/13) = \alpha$, so $\sin\alpha = 5/13$, $0 < \alpha < \pi/2$.

$\cos\alpha = \sqrt{(1 - 25/169)} = 12/13$, so $\tan\alpha = 5/12$ and $\alpha = \tan^{-1}(5/12)$.

L.H.S. $= \tan^{-1}(5/12) + \tan^{-1}(7/17)$

$= \tan^{-1}[(5/12 + 7/17) / (1 - (5/12)(7/17))]$ (since $(5/12)(7/17) < 1$)

$= \tan^{-1}[((85 + 84)/204) / ((204 - 35)/204)]$

$= \tan^{-1}(169/169) = \tan^{-1}(1) = \pi/4$ = R.H.S.

Marking: Converting $\sin^{-1}(5/13)$ to $\tan^{-1}(5/12)$ – 1 mark · Using the $\tan^{-1}x + \tan^{-1}y$ formula – 1 mark · Simplifying to $\pi/4$ – 1 mark

Examiner tip: State the condition $xy < 1$ before using $\tan^{-1}x + \tan^{-1}y = \tan^{-1}[(x + y)/(1 - xy)]$.

Q.18 Show that the equation $2x^2 + 3xy - 2y^2 + x + 7y - 3 = 0$ represents a pair of straight lines. Also find the angle between them.

Answer (3 marks)

Comparing with $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$:

$a = 2$, $h = 3/2$, $b = -2$, $g = 1/2$, $f = 7/2$, $c = -3$.

The condition is $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$.

$$\begin{vmatrix} 2 & 3/2 & 1/2 \\ 3/2 & -2 & 7/2 \\ 1/2 & 7/2 & -3 \end{vmatrix} = 2(6 - 49/4) - (3/2)(-9/2 - 7/4) + (1/2)(21/4 + 1)$$

$= 2(-25/4) - (3/2)(-25/4) + (1/2)(25/4)$

$= -50/4 + 75/8 + 25/8 = -100/8 + 100/8 = 0$

$\therefore$ The equation represents a pair of straight lines.

Also $h^2 - ab = 9/4 + 4 > 0$.

Angle: $a + b = 2 + (-2) = 0$, so the lines are perpendicular.

$\therefore$ **Angle between the lines = 90°.**

(Indeed, the equation factorises as $(x + 2y - 1)(2x - y + 3) = 0$.)

Marking: Identifying a, b, c, f, g, h – ½ mark · Determinant equal to 0 – 1½ marks · a + b = 0, so angle 90° – 1 mark

Examiner tip: Halve the coefficients of xy, x and y to get h, g and f. Using 3, 1, 7 instead of 3/2, 1/2, 7/2 is the most common error.

Q.19 Find λ if the points A(3, 2, 1), B(4, λ, 5), C(4, 2, −2) and D(6, 5, −1) are coplanar.

Answer (3 marks)

$\bar{AB} = (1, \lambda - 2, 4)$, $\bar{AC} = (1, 0, -3)$, $\bar{AD} = (3, 3, -2)$.

The points are coplanar $\Leftrightarrow [\bar{AB}\ \bar{AC}\ \bar{AD}] = 0$.

$$\begin{vmatrix} 1 & \lambda - 2 & 4 \\ 1 & 0 & -3 \\ 3 & 3 & -2 \end{vmatrix} = 0$$

$\therefore 1(0 + 9) - (\lambda - 2)(-2 + 9) + 4(3 - 0) = 0$

$\therefore 9 - 7(\lambda - 2) + 12 = 0$

$\therefore 35 - 7\lambda = 0$

$\therefore$ **$\lambda = 5$**

Marking: Three vectors from A – 1 mark · Scalar triple product determinant = 0 – 1 mark · λ = 5 – 1 mark

Examiner tip: Use position vectors relative to the same point (A) for all three vectors.

Q.20 Find the shortest distance between the lines
$\bar{r} = (\hat{i} + 2\hat{j} + \hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k})$ and $\bar{r} = (2\hat{i} - \hat{j} - \hat{k}) + \mu(2\hat{i} + \hat{j} + 2\hat{k})$.

Answer (3 marks)

$\bar{a}_1 = \hat{i} + 2\hat{j} + \hat{k}$, $\bar{b}_1 = \hat{i} - \hat{j} + \hat{k}$; $\bar{a}_2 = 2\hat{i} - \hat{j} - \hat{k}$, $\bar{b}_2 = 2\hat{i} + \hat{j} + 2\hat{k}$.

$\bar{a}_2 - \bar{a}_1 = \hat{i} - 3\hat{j} - 2\hat{k}$

$$\bar{b}_1 \times \bar{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ 2 & 1 & 2 \end{vmatrix} = \hat{i}(-2 - 1) - \hat{j}(2 - 2) + \hat{k}(1 + 2) = -3\hat{i} + 3\hat{k}$$

$|\bar{b}_1 \times \bar{b}_2| = \sqrt{(9 + 9)} = 3\sqrt{2}$

$(\bar{a}_2 - \bar{a}_1) \cdot (\bar{b}_1 \times \bar{b}_2) = (1)(-3) + (-3)(0) + (-2)(3) = -9$

$d = |-9| \,/\, (3\sqrt{2}) = 3/\sqrt{2}$

$\therefore$ **Shortest distance = $3/\sqrt{2} = (3\sqrt{2})/2$ units**

Marking: $\bar{b}_1 \times \bar{b}_2$ and its magnitude – 1 mark · $(\bar{a}_2 - \bar{a}_1) \cdot (\bar{b}_1 \times \bar{b}_2)$ – 1 mark · Final distance – 1 mark

Examiner tip: Take the modulus at the end; a distance can never be negative.

Q.21 A small workshop makes desk lamps and wall lamps. A desk lamp needs 2 hours of assembly and 1 hour of finishing; a wall lamp needs 1 hour of assembly and 1 hour of finishing. In a week, at most 40 hours of assembly and 30 hours of finishing are available. The profit is ₹40 on a desk lamp and ₹30 on a wall lamp. Formulate the problem as an L.P.P. to maximise the profit.

Answer (3 marks)

Let x = number of desk lamps and y = number of wall lamps made per week.

	Desk lamp (x)	Wall lamp (y)	Hours available
Assembly	2	1	40
Finishing	1	1	30
Profit (₹)	40	30	

Total profit $z = 40x + 30y$.

The L.P.P. is:

Maximise $z = 40x + 30y$

subject to $2x + y \le 40$ (assembly), $x + y \le 30$ (finishing), $x \ge 0$, $y \ge 0$.

Marking: Defining the variables – ½ mark · Objective function – 1 mark · Constraints (including non-negativity) – 1½ marks

Examiner tip: Always define x and y in words and write $x \ge 0$, $y \ge 0$; the non-negativity constraints carry marks.

Q.22 If $y = e^{2x} \sin x$, show that $d^2y/dx^2 - 4\,dy/dx + 5y = 0$.

Answer (3 marks)

$y = e^{2x} \sin x$

$dy/dx = 2e^{2x} \sin x + e^{2x} \cos x = e^{2x}(2 \sin x + \cos x)$

$d^2y/dx^2 = 2e^{2x}(2 \sin x + \cos x) + e^{2x}(2 \cos x - \sin x) = e^{2x}(3 \sin x + 4 \cos x)$

L.H.S. $= e^{2x}[(3 \sin x + 4 \cos x) - 4(2 \sin x + \cos x) + 5 \sin x]$

$= e^{2x}[3 \sin x + 4 \cos x - 8 \sin x - 4 \cos x + 5 \sin x] = e^{2x} \times 0 = 0 =$ R.H.S.

Marking: dy/dx – 1 mark · d^2y/dx^2 – 1 mark · Substitution and simplification – 1 mark

Examiner tip: Use the product rule at both stages and take e^{2x} common to keep the algebra short.

Q.23 Find the intervals in which $f(x) = 2x^3 - 21x^2 + 60x + 5$ is (a) increasing (b) decreasing.

Answer (3 marks)

$f'(x) = 6x^2 - 42x + 60 = 6(x^2 - 7x + 10) = 6(x - 2)(x - 5)$

(a) f is increasing when $f'(x) > 0$, i.e. $(x - 2)(x - 5) > 0 \Rightarrow x < 2$ or $x > 5$.

$\therefore$ **f is increasing on $(-\infty, 2) \cup (5, \infty)$.**

(b) f is decreasing when $f'(x) < 0$, i.e. $(x - 2)(x - 5) < 0 \Rightarrow 2 < x < 5$.

$\therefore$ **f is decreasing on (2, 5).**

Marking: $f'(x)$ factorised – 1 mark · Increasing interval – 1 mark · Decreasing interval – 1 mark

Examiner tip: For a product $(x - 2)(x - 5)$, '> 0' means outside the roots and '< 0' means between them.

Q.24 Complete the following activity to evaluate $\int e^x (x^2 + 5x + 5)/(x + 5)^2\,dx$.

Activity: $(x^2 + 5x + 5)/(x + 5)^2 = x(x + 5)/(x + 5)^2 + 5/(x + 5)^2 = x/(x + 5) +$ ☐

Let $f(x) =$ ☐, then $f'(x) = 5/(x + 5)^2$

$I = \int e^x[f(x) + f'(x)]\,dx =$ ☐ $+ c$

$\therefore I =$ ☐ $+ c$

Answer (3 marks)

$(x^2 + 5x + 5)/(x + 5)^2 = x/(x + 5) +$ **$5/(x + 5)^2$**

$f(x) =$ **$x/(x + 5)$**; check: $f'(x) = [(x + 5)(1) - x(1)]/(x + 5)^2 = 5/(x + 5)^2$

$I = \int e^x[f(x) + f'(x)]\,dx = \mathbf{e^x f(x)} + c$

$\therefore I = \mathbf{xe^x/(x + 5)} + c$

Marking: Splitting the fraction and $f(x) = x/(x + 5)$ – 1 mark · Standard result $e^x f(x) + c$ – 1 mark · Final answer – 1 mark

Examiner tip: Check by differentiating $xe^x/(x + 5)$: you must get back $e^x (x^2 + 5x + 5)/(x + 5)^2$.

Q.25 Find the area of the region bounded by the parabola $y = x^2$ and the line $y = 2x + 3$.

Answer (3 marks)

Points of intersection: $x^2 = 2x + 3 \Rightarrow x^2 - 2x - 3 = 0 \Rightarrow (x - 3)(x + 1) = 0 \Rightarrow x = -1, 3$.

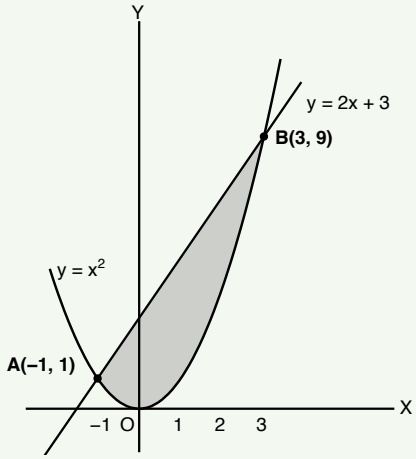
Points A(−1, 1) and B(3, 9). Between them the line lies above the parabola (see figure).

Required area $= \int_{-1}^{3} [(2x + 3) - x^2]\,dx = [x^2 + 3x - x^3/3]_{-1}^{3}$

$= (9 + 9 - 9) - (1 - 3 + 1/3)$

$= 9 - (-5/3) = 27/3 + 5/3$

$\therefore$ **Area = 32/3 sq. units**



Marking: Points of intersection (with rough sketch) – 1 mark · Correct integral set-up – 1 mark · Evaluation 32/3 sq. units – 1 mark

Examiner tip: Draw a rough sketch to decide which curve is on top, and write the unit 'sq. units' with the answer.

Q.26 Solve the differential equation $dy/dx + 3y/x = x$.

Answer (3 marks)

This is linear: $dy/dx + Py = Q$ with $P = 3/x$, $Q = x$.

I.F. $= e^{\int P\,dx} = e^{3 \log x} = e^{\log x^3} = x^3$

Solution: $y \cdot (\text{I.F.}) = \int Q \cdot (\text{I.F.})\,dx + c$

$\therefore x^3 y = \int x^4\,dx + c = x^5/5 + c$

$\therefore \mathbf{x^3 y = x^5/5 + c}$, i.e. $y = x^2/5 + c/x^3$.

Marking: Identifying P, Q and I.F. $= x^3$ – 1 mark · Correct solution formula – 1 mark · Integration and final answer – 1 mark

Examiner tip: Write the general solution with the arbitrary constant c; omitting it costs a mark.

SECTION D Attempt any FIVE of the following questions (Q.27 to Q.34, 4 marks each). **[20]**

Q.27 Solve the following equations by the method of reduction:
$x + y + z = 4,\ 2x - y + z = 8,\ x + 3y - 2z = -7$

Answer (4 marks)

The equations in matrix form AX = B are

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 1 & 3 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ -7 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - R_1$ (applied to both sides):

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & -1 \\ 0 & 2 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ -11 \end{bmatrix}$$

$R_3 \to 3R_3 + 2R_2$:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & -1 \\ 0 & 0 & -11 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ -33 \end{bmatrix}$$

A is now upper triangular. By equality of matrices,

$x + y + z = 4,\ -3y - z = 0,\ -11z = -33$

$\therefore z = 3;\ -3y = 3 \Rightarrow y = -1;\ x = 4 + 1 - 3 = 2$

$\therefore$ **x = 2, y = −1, z = 3**

Check: $2(2) - (-1) + 3 = 8$ ✓; $2 + 3(-1) - 2(3) = -7$ ✓

Marking: Matrix form AX = B – ½ mark · First reduction step – 1 mark · Upper triangular form – 1 mark · Back substitution and values of x, y, z – 1½ marks

Examiner tip: Apply each row operation to both sides (A and B) at the same time. Checking the answer in one equation catches most slips.

Q.28 Solve the following L.P.P. graphically:
Maximise $z = 3x + 4y$ subject to $2x + y \le 16,\ x + 2y \le 14,\ x \ge 0,\ y \ge 0$.

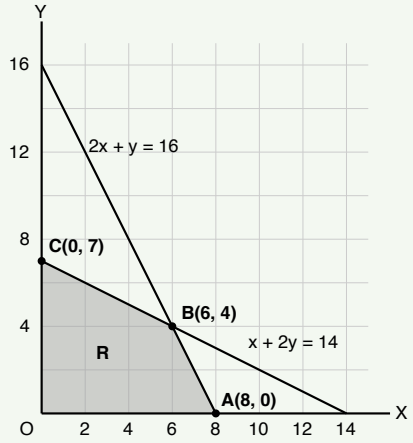
Answer (4 marks)

Draw the lines $2x + y = 16$ (through (8, 0) and (0, 16)) and $x + 2y = 14$ (through (14, 0) and (0, 7)). The origin satisfies both inequalities, so each region lies on the origin side. With $x \ge 0,\ y \ge 0$ the feasible region is the shaded quadrilateral OABC (see figure).

B is the intersection of $2x + y = 16$ and $x + 2y = 14$: $y = 16 - 2x \Rightarrow x + 32 - 4x = 14 \Rightarrow x = 6,\ y = 4$. So B(6, 4).

Corner point	z = 3x + 4y
O(0, 0)	0
A(8, 0)	24
B(6, 4)	34
C(0, 7)	28

$\therefore$ **z is maximum = 34 at x = 6, y = 4.**



Marking: Drawing both lines correctly – 1 mark · Shading the feasible region – 1 mark · Corner points including B(6, 4) – 1 mark · Maximum value z = 34 at (6, 4) – 1 mark

Examiner tip: Only a rough sketch is expected, but label the corner points and the lines. Evaluate z at every corner of the feasible region.

Q.29 Find the vector and Cartesian equations of the plane passing through the points A(1, 2, 0), B(2, 0, 1) and C(0, 1, 3).

Answer (4 marks)

$\bar{AB} = (1, -2, 1)$, $\bar{AC} = (-1, -1, 3)$

$$\text{Normal } \bar{n} = \bar{AB} \times \bar{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 1 \\ -1 & -1 & 3 \end{vmatrix} = \hat{i}(-6+1) - \hat{j}(3+1) + \hat{k}(-1-2) = -5\hat{i} - 4\hat{j} - 3\hat{k}$$

Taking the normal as $\bar{n} = 5\hat{i} + 4\hat{j} + 3\hat{k}$ and $\bar{a} = \hat{i} + 2\hat{j}$ (point A):

$\bar{a} \cdot \bar{n} = 5 + 8 + 0 = 13$

Vector equation: $\bar{r} \cdot (5\hat{i} + 4\hat{j} + 3\hat{k}) = 13$

Cartesian equation: $5x + 4y + 3z = 13$

Check: B: 10 + 0 + 3 = 13 ✓; C: 0 + 4 + 9 = 13 ✓

Marking: Two vectors in the plane – 1 mark · Normal vector by cross product – 1 mark · Vector equation – 1 mark · Cartesian equation – 1 mark

Examiner tip: Substitute all three points in your final equation; it takes a few seconds and confirms the normal.

Q.30 An open box is to be made from a rectangular sheet of tin 30 cm × 14 cm by cutting equal squares from the four corners and folding up the sides. Find the side of the square to be cut off so that the volume of the box is maximum. Also find the maximum volume.

Answer (4 marks)

Let the side of each square be x cm ($0 < x < 7$).

Box: length $(30 - 2x)$, breadth $(14 - 2x)$, height x.

$V = x(30 - 2x)(14 - 2x) = 4x^3 - 88x^2 + 420x$

$dV/dx = 12x^2 - 176x + 420 = 4(3x^2 - 44x + 105) = 4(3x - 35)(x - 3)$

$dV/dx = 0 \Rightarrow x = 3$ or $x = 35/3$. Since $x < 7$, $x = 35/3$ is rejected; so $x = 3$.

$d^2V/dx^2 = 24x - 176$; at $x = 3$ it is $72 - 176 = -104 < 0$, so V is maximum at $x = 3$.

$V_{max} = 3 \times 24 \times 8 = 576$

$\therefore$ **Side of square = 3 cm; maximum volume = 576 cm^3**

Marking: Expression for V – 1 mark · dV/dx = 0 and x = 3 (rejecting 35/3) – 1 mark · Second derivative test – 1 mark · Maximum volume 576 cm^3 – 1 mark

Examiner tip: Give a reason for rejecting x = 35/3 (the breadth 14 − 2x would be negative) and show the second derivative test.

Q.31 Complete the following activity to evaluate $I = \int_2^6 \sqrt{(x+1)} / (\sqrt{(x+1)} + \sqrt{(9-x)})\, dx$.

Activity: Using $\int_a^b f(x)\, dx = \int_a^b f(a+b-x)\, dx$, with $a + b = \square$,

$I = \int_2^6 \sqrt{(\square)} / (\sqrt{(9-x)} + \sqrt{(x+1)})\, dx$ ...(2)

Adding I and (2): $2I = \int_2^6 \square\, dx = [x]_2^6 = \square$

$\therefore I = \square$

Answer (4 marks)

$a + b = 2 + 6 =$ **8**, so x is replaced by $8 - x$: $x + 1$ becomes $9 - x$ and $9 - x$ becomes $x + 1$.

$I = \int_2^6 \sqrt{(\mathbf{9-x})} / (\sqrt{(9-x)} + \sqrt{(x+1)})\, dx$ ...(2)

$2I = \int_2^6 [\sqrt{(x+1)} + \sqrt{(9-x)}] / [\sqrt{(x+1)} + \sqrt{(9-x)}]\, dx = \int_2^6 \mathbf{1}\, dx = [x]_2^6 = \mathbf{4}$

$\therefore I = \mathbf{2}$

Marking: $a + b = 8$ and correct use of the property – 1 mark · Integral (2) – 1 mark · $2I = \int 1\, dx = 4$ – 1 mark · $I = 2$ – 1 mark

Examiner tip: The property uses $a + b - x$, not $b - x$, when the lower limit is not 0. Check: $(8 - x) + 1 = 9 - x$ and $9 - (8 - x) = x + 1$.

Q.32 Evaluate: $\int x^2 / [(x^2+1)(x^2+9)]\, dx$

Answer (4 marks)

Put $x^2 = t$ only for the partial fractions:

$t / [(t+1)(t+9)] = A/(t+1) + B/(t+9) \Rightarrow t = A(t+9) + B(t+1)$

$t = -1$: $-1 = 8A \Rightarrow A = -1/8$; $t = -9$: $-9 = -8B \Rightarrow B = 9/8$

$\therefore x^2 / [(x^2+1)(x^2+9)] = (-1/8)\cdot 1/(x^2+1) + (9/8)\cdot 1/(x^2+9)$

$I = (-1/8)\int dx/(x^2+1) + (9/8)\int dx/(x^2+3^2)$

$= (-1/8)\tan^{-1}x + (9/8)(1/3)\tan^{-1}(x/3) + c$

$\therefore$ **$I = (3/8)\tan^{-1}(x/3) - (1/8)\tan^{-1}x + c$**

Marking: Partial fraction form – 1 mark · Values $A = -1/8$, $B = 9/8$ – 1 mark · Standard integrals – 1 mark · Final answer with c – 1 mark

Examiner tip: Do not integrate with respect to t: $x^2 = t$ is used only to split the fraction; the integration is still in x.

Q.33 The number of bacteria in a culture increases at a rate proportional to the number present. There are 800 bacteria at the start and 1200 after 3 hours. How many bacteria will there be after 9 hours?

Answer (4 marks)

Let N be the number at time t hours. $dN/dt \propto N \Rightarrow dN/dt = kN$.

$\int dN/N = \int k\, dt \Rightarrow \log N = kt + c$

At $t = 0$, $N = 800 \Rightarrow c = \log 800$. $\therefore \log(N/800) = kt$ ...(1)

At $t = 3$, $N = 1200 \Rightarrow \log(3/2) = 3k \Rightarrow k = (1/3)\log(3/2)$

$\therefore \log(N/800) = (t/3)\log(3/2) \Rightarrow N = 800\,(3/2)^{t/3}$

At $t = 9$: $N = 800\,(3/2)^3 = 800 \times 27/8 = 2700$

$\therefore$ **After 9 hours there will be 2700 bacteria.**

Marking: Forming $dN/dt = kN$ – 1 mark · Solving: $\log N = kt + c$ and using $N(0) = 800$ – 1 mark · Finding k from $N(3) = 1200$ – 1 mark · $N(9) = 2700$ – 1 mark

Examiner tip: Keep k as $(1/3)\log(3/2)$; there is no need to find its decimal value, and log tables are not needed here.

Q.34 A test has 6 multiple choice questions, each with 4 alternatives of which exactly one is correct. A student guesses the answer to every question. Find the probability that the student gets (a) exactly 2 answers correct (b) at least one answer correct.

Answer (4 marks)

Let X = number of correct answers. X ~ B(n = 6, p = 1/4), q = 3/4.

$P(X = x) = {}^{6}C_{x}\,(1/4)^{x}\,(3/4)^{6-x}$, x = 0, 1, …, 6

(a) $P(X = 2) = {}^{6}C_{2}\,(1/4)^{2}\,(3/4)^{4} = 15 \times (1/16) \times (81/256) =$ **1215/4096**

(b) $P(X \geq 1) = 1 - P(X = 0) = 1 - (3/4)^{6} = 1 - 729/4096 =$ **3367/4096**

Marking: Identifying n, p, q and the p.m.f. – 1 mark · P(X = 2) – 1½ marks · P(X ≥ 1) using the complement – 1½ marks

Examiner tip: For ‘at least one’ use 1 − P(X = 0); adding six terms wastes time and invites errors.

All the best for your exam!