Mathematics (041)

CBSE Class 12 Board Examination 2027 – Practice Paper No. 2

Model Answer Paper

Time: 3 hours **Max. Marks: 80**

Model answers show one correct method; any other correct method gets full marks.

SECTION A This section comprises 18 multiple choice questions (Q1–Q18) of 1 mark each. Choose the correct option. **[18]**

1. The relation R on the set A = {1, 2, 3} given by R = {(1, 1), (2, 2), (3, 3), (1, 3), (3, 1), (2, 3)} is

(A) reflexive and symmetric but not transitive
(B) an equivalence relation
(C) reflexive but neither symmetric nor transitive
(D) reflexive and transitive but not symmetric

Answer (1 mark)
(C) reflexive but neither symmetric nor transitive
(1, 1), (2, 2), (3, 3) ∈ R, so R is reflexive.
(2, 3) ∈ R but (3, 2) ∉ R, so R is not symmetric.
(2, 3) ∈ R and (3, 1) ∈ R but (2, 1) ∉ R, so R is not transitive.
Marking: Correct option – 1 mark

2. The range of the principal value branch of the function $y = \sec^{-1} x$ is

(A) $[0, \pi] - \{\pi/2\}$
(B) $[-\pi/2, \pi/2] - \{0\}$
(C) $(0, \pi)$
(D) $[-\pi/2, \pi/2]$

Answer (1 mark)
(A) $[0, \pi] - \{\pi/2\}$
$\sec^{-1} : R - (-1, 1) \to [0, \pi] - \{\pi/2\}$ is the principal value branch ($\pi/2$ is left out because sec ($\pi/2$) is not defined). $[-\pi/2, \pi/2] - \{0\}$ is the range for cosec^{-1}.
Marking: Correct option – 1 mark

3. If $A = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$ and $A^4 = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$, then the value of k is

(A) 81
(B) 12
(C) 4
(D) 7

Answer (1 mark)
(B) 12
$A^2 = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 6 & 1 \end{bmatrix}$, $A^4 = A^2A^2 = \begin{bmatrix} 1 & 0 \\ 6 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 6 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 12 & 1 \end{bmatrix}$. So k = 12 (not 3^4 = 81: matrix powers are not taken entry by entry).
Marking: Correct option – 1 mark

4. If the matrix $\begin{bmatrix} 0 & a & -3 \\ 2 & 0 & -1 \\ b & 1 & 0 \end{bmatrix}$ is skew-symmetric, then the value of a + b is

(A) 5
(B) −1
(C) −5
(D) 1

Answer (1 mark)
(D) 1
For a skew-symmetric matrix, $a_{ij} = -a_{ji}$. So $a = a_{12} = -a_{21} = -2$ and $b = a_{31} = -a_{13} = 3$ (also $a_{23} = -1 = -a_{32}$ ✓). Hence $a + b = -2 + 3 = 1$.
Marking: Correct option – 1 mark

5. The cofactor of the element a_{23} of the determinant $\begin{vmatrix} 1 & 3 & -2 \\ 4 & -5 & 6 \\ 3 & 5 & 2 \end{vmatrix}$ is

(A) 4 (B) −4
(C) 14 (D) −14

Answer (1 mark)
(A) 4
Minor M_{23} (delete row 2 and column 3) $= \begin{vmatrix} 1 & 3 \\ 3 & 5 \end{vmatrix} = 5 - 9 = -4$.
Cofactor $A_{23} = (-1)^{2+3} M_{23} = -(-4) = 4$.
Marking: Correct option – 1 mark

6. The function $f(x) = |x - 3| + 2$, $x \in R$, is

(A) continuous and differentiable at $x = 3$ (B) continuous but not differentiable at $x = 3$
(C) differentiable but not continuous at $x = 3$ (D) neither continuous nor differentiable at $x = 3$

Answer (1 mark)
(B) continuous but not differentiable at $x = 3$
Limit from both sides = 2 = f(3), so f is continuous at 3.
LHD = −1 ($f(x) = 5 - x$ for $x < 3$) and RHD = 1 ($f(x) = x - 1$ for $x > 3$); LHD ≠ RHD, so f is not differentiable at $x = 3$.
Marking: Correct option – 1 mark

7. A cube-shaped block of ice is melting so that each edge decreases at the rate of 0.4 cm/min. The rate at which its total surface area decreases when the edge is 15 cm is

(A) 36 cm^2/min (B) 180 cm^2/min
(C) 72 cm^2/min (D) 144 cm^2/min

Answer (1 mark)
(C) 72 cm^2/min
$S = 6x^2 \Rightarrow dS/dt = 12x\,(dx/dt) = 12 \times 15 \times (-0.4) = -72$. The surface area decreases at 72 cm^2/min.
Marking: Correct option – 1 mark

8. $\int e^x (\tan x + \sec^2 x)\,dx$ is equal to

(A) $e^x \tan x + C$ (B) $e^x \sec^2 x + C$
(C) $e^x \sec x + C$ (D) $e^x (\tan x + \sec x) + C$

Answer (1 mark)
(A) $e^x \tan x + C$
The integrand is $e^x[f(x) + f'(x)]$ with $f(x) = \tan x$, $f'(x) = \sec^2 x$, and $\int e^x[f(x) + f'(x)]\,dx = e^x f(x) + C$.
Marking: Correct option – 1 mark

9. If $\int_0^a (3x^2 + 2x + 1)\,dx = 14$, where a is real, then a is equal to

(A) 1 (B) 3

(C) −2 (D) 2

Answer (1 mark)

(D) 2

$[x^3 + x^2 + x]_0^a = a^3 + a^2 + a = 14 \Rightarrow (a - 2)(a^2 + 3a + 7) = 0$. $a^2 + 3a + 7 = 0$ has no real root, so a = 2.

Marking: Correct option – 1 mark

10. The area of the region bounded by the line $y = 2x - 4$, the x-axis and the lines $x = 0$ and $x = 4$ is

(A) 0 sq. units (B) 4 sq. units

(C) 8 sq. units (D) 16 sq. units

Answer (1 mark)

(C) 8 sq. units

The line cuts the x-axis at $x = 2$; the region is below the axis for $0 \le x \le 2$ and above it for $2 \le x \le 4$.

Area $= \left|\int_0^2 (2x - 4)\,dx\right| + \int_2^4 (2x - 4)\,dx = |-4| + 4 = 8$. ($\int_0^4 (2x - 4)\,dx = 0$ is not the area.)

Marking: Correct option – 1 mark

11. The solution of the differential equation $dy/dx = 2xy$ that satisfies $y(0) = 3$ is

(A) $y = 3e^{2x}$ (B) $y = 3e^{x^2}$

(C) $y = e^{x^2} + 2$ (D) $y = 3 + x^2$

Answer (1 mark)

(B) $y = 3e^{x^2}$

Separating the variables: $dy/y = 2x\,dx \Rightarrow \log|y| = x^2 + C \Rightarrow y = Ae^{x^2}$. $y(0) = 3 \Rightarrow A = 3$.

Marking: Correct option – 1 mark

12. If $|\vec{a}| = 3$, $|\vec{b}| = 4$ and $|\vec{a} + \vec{b}| = 5$, then $|\vec{a} - \vec{b}|$ is equal to

(A) 5 (B) 1

(C) 7 (D) $\sqrt{7}$

Answer (1 mark)

(A) 5

$|\vec{a} + \vec{b}|^2 = 9 + 16 + 2\,\vec{a} \cdot \vec{b} = 25 \Rightarrow \vec{a} \cdot \vec{b} = 0$.

$|\vec{a} - \vec{b}|^2 = 9 + 16 - 0 = 25 \Rightarrow |\vec{a} - \vec{b}| = 5$.

Marking: Correct option – 1 mark

13. If $|\vec{a}| = 2$, $|\vec{b}| = 5$ and $|\vec{a} \times \vec{b}| = 8$, then $|\vec{a} \cdot \vec{b}|$ is equal to

(A) 10 (B) 3

(C) 2 (D) 6

Answer (1 mark)

(D) 6

$|\vec{a} \times \vec{b}|^2 + (\vec{a} \cdot \vec{b})^2 = |\vec{a}|^2|\vec{b}|^2$ (since $\sin^2\theta + \cos^2\theta = 1$) $\Rightarrow 64 + (\vec{a} \cdot \vec{b})^2 = 100 \Rightarrow |\vec{a} \cdot \vec{b}| = 6$.

Marking: Correct option – 1 mark

14. The angle between the lines $(x + 1)/2 = (y - 3)/(-1) = (z - 2)/1$ and $x/1 = (y + 4)/1 = (z - 5)/2$ is

(A) $\pi/6$ (B) $\pi/4$

(C) $\pi/3$ (D) $\pi/2$

Answer (1 mark)

(C) $\pi/3$

Direction ratios: $2, -1, 1$ and $1, 1, 2$. $\cos\theta = |2 - 1 + 2|/(\sqrt{6} \cdot \sqrt{6}) = 3/6 = 1/2 \Rightarrow \theta = \pi/3$.

Marking: Correct option – 1 mark

15. The corner points of the bounded feasible region of a linear programming problem are (0, 0), (5, 0), (4, 3) and (0, 6). The maximum value of the objective function $Z = 3x + 4y$ occurs at

(A) (4, 3) only (B) (0, 6) only

(C) every point of the line segment joining (4, 3) and (0, 6) (D) (5, 0)

Answer (1 mark)

(C) every point of the line segment joining (4, 3) and (0, 6)

$Z(0, 0) = 0$, $Z(5, 0) = 15$, $Z(4, 3) = 24$, $Z(0, 6) = 24$. The maximum 24 occurs at two adjacent corners, so it occurs at every point of the segment joining them.

Marking: Correct option – 1 mark

16. Which of the following points does NOT lie in the feasible region determined by the constraints $2x + 3y \le 12$, $x + y \ge 2$, $x \ge 0$, $y \ge 0$?

(A) (1, 1) (B) (0, 1)

(C) (3, 2) (D) (6, 0)

Answer (1 mark)

(B) (0, 1)

For (0, 1): $x + y = 1 < 2$, so the constraint $x + y \ge 2$ fails. (1, 1): $5 \le 12$, $2 \ge 2$ ✓; (3, 2): $12 \le 12$, $5 \ge 2$ ✓; (6, 0): $12 \le 12$, $6 \ge 2$ ✓.

Marking: Correct option – 1 mark

17. If $P(A) = 2/5$, $P(B) = 1/2$ and $P(B|A) = 3/4$, then $P(A|B)$ is equal to

(A) 3/10 (B) 3/4

(C) 2/5 (D) 3/5

Answer (1 mark)

(D) 3/5

$P(A \cap B) = P(A)\,P(B|A) = (2/5)(3/4) = 3/10$. $P(A|B) = P(A \cap B)/P(B) = (3/10)/(1/2) = 3/5$.

Marking: Correct option – 1 mark

18. A bag contains 4 red and 6 black balls. Two balls are drawn at random one after the other without replacement. The probability that both balls are red is

(A) 4/25 (B) 2/15

(C) 4/15 (D) 1/5

Answer (1 mark)

(B) 2/15

By the multiplication theorem, $P = (4/10) \times (3/9) = 12/90 = 2/15$. (4/25 would be the answer with replacement.)

Marking: Correct option – 1 mark

SECTION A (contd.) Assertion–Reason based questions (Q19–Q20), 1 mark each. Two statements are given, one labelled Assertion (A) and the other labelled Reason (R). Select the correct answer from the options (A), (B), (C) and (D) given below each question. **[2]**

19. **Assertion (A):** $\tan^{-1}(\tan(4\pi/5)) = -\pi/5$.

Reason (R): The domain of the function $\tan^{-1}$ is the set R of all real numbers.

(A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of Assertion (A).

(B) Both Assertion (A) and Reason (R) are true but Reason (R) is not the correct explanation of Assertion (A).

(C) Assertion (A) is true but Reason (R) is false.

(D) Assertion (A) is false but Reason (R) is true.

Answer (1 mark)

(B) Both A and R are true but R is not the correct explanation of A.

A: $\tan(4\pi/5) = \tan(4\pi/5 - \pi) = \tan(-\pi/5)$, and $-\pi/5$ lies in the principal value branch $(-\pi/2, \pi/2)$, so $\tan^{-1}(\tan(4\pi/5)) = -\pi/5$. True.

R: $\tan^{-1}$ is defined for every real number. True.

But A follows from the range $(-\pi/2, \pi/2)$ of the principal value branch, not from the domain, so R does not explain A.

Marking: Correct option – 1 mark

20. **Assertion (A):** If A and B are independent events with P(A) = 0.4 and P(B) = 0.5, then $P(A \cup B) = 0.7$.

Reason (R): For any two events A and B, $P(A \cap B) = P(A) \cdot P(B)$.

(A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of Assertion (A).

(B) Both Assertion (A) and Reason (R) are true but Reason (R) is not the correct explanation of Assertion (A).

(C) Assertion (A) is true but Reason (R) is false.

(D) Assertion (A) is false but Reason (R) is true.

Answer (1 mark)

(C) A is true but R is false.

A: $P(A \cap B) = 0.4 \times 0.5 = 0.2$, so $P(A \cup B) = 0.4 + 0.5 - 0.2 = 0.7$. True.

R: $P(A \cap B) = P(A)\,P(B)$ holds only when A and B are independent, not for any two events. False.

Marking: Correct option – 1 mark

SECTION B This section comprises 5 very short answer (VSA) type questions of 2 marks each. **[10]**

21. Find the domain of the function $f(x) = \sin^{-1}(x^2 - 3)$.

Answer (2 marks)

$\sin^{-1} t$ is defined only for $-1 \le t \le 1$. So we need $-1 \le x^2 - 3 \le 1$

$\Rightarrow 2 \le x^2 \le 4$.

$x^2 \le 4 \Rightarrow -2 \le x \le 2$, and $x^2 \ge 2 \Rightarrow x \le -\sqrt{2}$ or $x \ge \sqrt{2}$.

Hence the domain is $\mathbf{[-2, -\sqrt{2}] \cup [\sqrt{2}, 2]}$.

Marking: Writing $-1 \le x^2 - 3 \le 1$, i.e. $2 \le x^2 \le 4$ – 1 mark · Domain $[-2, -\sqrt{2}] \cup [\sqrt{2}, 2]$ – 1 mark

Examiner tip: $x^2 \ge 2$ gives TWO pieces ($x \le -\sqrt{2}$ or $x \ge \sqrt{2}$). Writing only $[\sqrt{2}, 2]$ loses the second mark.

22. (A) If $x^2 + xy + y^3 = 7$, find dy/dx at the point (2, 1).

OR

(B) If $x = t^2 + 2t$ and $y = t^3 - 3t$, find dy/dx in terms of t and its value at t = 3.

Answer (2 marks)

(A) Differentiating both sides with respect to x:

$2x + (y + x\,dy/dx) + 3y^2\,dy/dx = 0$

$\Rightarrow (x + 3y^2)\, dy/dx = -(2x + y) \Rightarrow dy/dx = -(2x + y)/(x + 3y^2)$.

(2, 1) lies on the curve $(4 + 2 + 1 = 7)$. At (2, 1): $dy/dx = -(4 + 1)/(2 + 3) = \mathbf{-1}$.

OR

(B) $dx/dt = 2t + 2$, $dy/dt = 3t^2 - 3$.

$dy/dx = (dy/dt)/(dx/dt) = 3(t^2 - 1)/(2(t + 1)) = \mathbf{3(t - 1)/2}$, $t \neq -1$.

At $t = 3$: $dy/dx = 3 \times 2/2 = \mathbf{3}$.

Marking: (A) Differentiating implicitly (product rule for xy) – 1 mark · (A) $dy/dx = -(2x + y)/(x + 3y^2)$ and value −1 – 1 mark · (B) dx/dt and dy/dt – 1 mark · (B) $dy/dx = 3(t - 1)/2$ and value 3 – 1 mark

Examiner tip: In (A), xy is a product: its derivative is y + x dy/dx. Students often write only x dy/dx.

23. The vectors $\vec{a} = 2\hat{i} + \hat{j} - 2\hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} + 2\hat{k}$ represent two adjacent sides of a parallelogram. Find the area of the parallelogram and show that its diagonals are perpendicular to each other.

Answer (2 marks)

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{vmatrix} = \hat{i}(2 + 4) - \hat{j}(4 + 2) + \hat{k}(4 - 1) = 6\hat{i} - 6\hat{j} + 3\hat{k}.$$

Area $= |\vec{a} \times \vec{b}| = \sqrt{(36 + 36 + 9)} = \sqrt{81} = \mathbf{9}$ **square units**.

The diagonals are $\vec{a} + \vec{b} = 3\hat{i} + 3\hat{j}$ and $\vec{a} - \vec{b} = \hat{i} - \hat{j} - 4\hat{k}$.

$(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 3 - 3 + 0 = 0$, so the diagonals are **perpendicular**. (This happens because $|\vec{a}| = |\vec{b}| = 3$: the parallelogram is a rhombus.)

Marking: $\vec{a} \times \vec{b}$ and area 9 sq. units – 1 mark · Diagonals and dot product = 0 – 1 mark

Examiner tip: Area of a parallelogram with ADJACENT SIDES $\vec{a}$, $\vec{b}$ is $|\vec{a} \times \vec{b}|$ (no ½). The ½ is used only when the diagonals are given.

24. (A) Find the vector and the cartesian equations of the line passing through the point (2, −1, 3) and parallel to the line $(x - 1)/3 = (y + 2)/(-2) = (4 - z)/6$.

OR

(B) Find the value of k for which the lines $(x + 2)/k = (y - 1)/3 = (z - 4)/(-2)$ and $(x - 3)/2 = y/(k + 1) = (z + 1)/4$ are perpendicular to each other.

Answer (2 marks)

(A) First write the given line in standard form: $(4 - z)/6 = (z - 4)/(-6)$. So the line is $(x - 1)/3 = (y + 2)/(-2) = (z - 4)/(-6)$, with direction ratios 3, −2, −6.

Required line (through (2, −1, 3), parallel to it):

Cartesian: $\mathbf{(x - 2)/3 = (y + 1)/(-2) = (z - 3)/(-6)}$

Vector: $\mathbf{\vec{r} = (2\hat{i} - \hat{j} + 3\hat{k}) + \lambda(3\hat{i} - 2\hat{j} - 6\hat{k})}$, $\lambda \in R$.

OR

(B) Direction ratios: k, 3, −2 and 2, k + 1, 4.

Lines are perpendicular $\Leftrightarrow a_1a_2 + b_1b_2 + c_1c_2 = 0$:

$2k + 3(k + 1) - 8 = 0 \Rightarrow 5k - 5 = 0 \Rightarrow \mathbf{k = 1}$.

Marking: (A) Direction ratios 3, −2, −6 (sign of z handled) – 1 mark · (A) Both equations – ½ + ½ mark · (B) Condition $a_1a_2 + b_1b_2 + c_1c_2 = 0$ – 1 mark · (B) k = 1 – 1 mark

Examiner tip: In (A), $(4 - z)/6$ must be rewritten as $(z - 4)/(-6)$ before reading the direction ratios; taking 3, −2, 6 is the most common error.

25. If $y = (a + bx)\, e^{2x}$, where a and b are constants, show that $d^2y/dx^2 - 4\, dy/dx + 4y = 0$.

Answer (2 marks)

$y = (a + bx)\, e^{2x}$

$dy/dx = b\, e^{2x} + 2(a + bx)\, e^{2x} = b\, e^{2x} + 2y$.

$d^2y/dx^2 = 2b\,e^{2x} + 2\,dy/dx = 2b\,e^{2x} + 2b\,e^{2x} + 4y = 4b\,e^{2x} + 4y.$

LHS $= (4b\,e^{2x} + 4y) - 4(b\,e^{2x} + 2y) + 4y = 4b\,e^{2x} + 4y - 4b\,e^{2x} - 8y + 4y = 0 =$ RHS. Hence proved.

Marking: $dy/dx = b\,e^{2x} + 2y$ – 1 mark · d^2y/dx^2 and substitution to get 0 – 1 mark

Examiner tip: Writing dy/dx as $b\,e^{2x} + 2y$ keeps the algebra short; differentiate the product $(a + bx)e^{2x}$ fully – do not forget the derivative of e^{2x} is $2e^{2x}$.

SECTION C This section comprises 6 short answer (SA) type questions of 3 marks each. **[18]**

26. Show that the function $f : R \to R$ defined by $f(x) = x^2 + 4x$ for $x \geq 0$ and $f(x) = 4x$ for $x < 0$ is both one-one and onto.

Answer (3 marks)

For $x \geq 0$, $f(x) = x(x + 4) \geq 0$; for $x < 0$, $f(x) = 4x < 0$. So a non-negative x and a negative x can never have the same image.

One-one: Let $f(x_1) = f(x_2)$. By the observation above, x_1 and x_2 are both ≥ 0 or both < 0.

If $x_1, x_2 \geq 0$: ${x_1}^2 + 4x_1 = {x_2}^2 + 4x_2 \Rightarrow (x_1 - x_2)(x_1 + x_2 + 4) = 0$. As $x_1 + x_2 + 4 \geq 4 > 0$, $x_1 = x_2$.

If $x_1, x_2 < 0$: $4x_1 = 4x_2 \Rightarrow x_1 = x_2$.

So f is one-one.

Onto: Let $y \in R$ (co-domain).

If $y \geq 0$, take $x = -2 + \sqrt{(4 + y)}$. Then $x \geq -2 + 2 = 0$, and $f(x) = x^2 + 4x = (x + 2)^2 - 4 = (4 + y) - 4 = y$.

If $y < 0$, take $x = y/4 < 0$: $f(x) = 4(y/4) = y$.

Every $y \in R$ has a pre-image, so f is onto.

Hence f is **one-one and onto (bijective)**.

Marking: Observation $f(x) \geq 0$ for $x \geq 0$ and $f(x) < 0$ for $x < 0$ – ½ mark · Proof of one-one (both cases) – 1½ marks · Proof of onto (pre-images $-2 + \sqrt{(4 + y)}$ for $y \geq 0$ and $y/4$ for $y < 0$) – 1 mark

Examiner tip: For $y \geq 0$, $x^2 + 4x - y = 0$ has two roots; take only $x = -2 + \sqrt{(4 + y)}$, the one that is ≥ 0. For $y < 0$ the pre-image must come from the piece $4x$.

27. Express the matrix $A = \begin{bmatrix} 3 & 5 & -1 \\ 1 & -2 & 4 \\ 7 & 0 & 6 \end{bmatrix}$ as the sum of a symmetric matrix and a skew-symmetric matrix.

Answer (3 marks)

$A' = \begin{bmatrix} 3 & 1 & 7 \\ 5 & -2 & 0 \\ -1 & 4 & 6 \end{bmatrix}$

$P = \tfrac{1}{2}(A + A') = \tfrac{1}{2}\begin{bmatrix} 6 & 6 & 6 \\ 6 & -4 & 4 \\ 6 & 4 & 12 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 3 \\ 3 & -2 & 2 \\ 3 & 2 & 6 \end{bmatrix}$

$P' = P$, so P is symmetric.

$Q = \tfrac{1}{2}(A - A') = \tfrac{1}{2}\begin{bmatrix} 0 & 4 & -8 \\ -4 & 0 & 4 \\ 8 & -4 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 & -4 \\ -2 & 0 & 2 \\ 4 & -2 & 0 \end{bmatrix}$

$Q' = -Q$, so Q is skew-symmetric.

$P + Q = \begin{bmatrix} 3 & 5 & -1 \\ 1 & -2 & 4 \\ 7 & 0 & 6 \end{bmatrix} = A.$

Hence **A = P + Q**, with P symmetric and Q skew-symmetric as above.

Marking: A' – ½ mark · $P = \tfrac{1}{2}(A + A')$ and showing $P' = P$ – 1 mark · $Q = \tfrac{1}{2}(A - A')$ and showing $Q' = -Q$ – 1 mark · Verifying $P + Q = A$ – ½ mark

Examiner tip: Check the diagonal of Q: every diagonal entry of a skew-symmetric matrix must be 0. If it is not, there is an arithmetic slip.

28. (A) Find: $\int (3x + 5)/\sqrt{(x^2 + 4x + 13)}\, dx$

OR

(B) Find: $\int \log (x^2 + 4)\, dx$

Answer (3 marks)

(A) $d/dx\,(x^2 + 4x + 13) = 2x + 4$. Write $3x + 5 = (3/2)(2x + 4) - 1$.

$I = (3/2) \int (2x + 4)/\sqrt{(x^2 + 4x + 13)}\, dx - \int dx/\sqrt{(x^2 + 4x + 13)}$

First integral: put $x^2 + 4x + 13 = t \Rightarrow (3/2) \int dt/\sqrt{t} = (3/2) \cdot 2\sqrt{t} = 3\sqrt{(x^2 + 4x + 13)}$.

Second: $x^2 + 4x + 13 = (x + 2)^2 + 3^2 \Rightarrow \int dx/\sqrt{((x + 2)^2 + 3^2)} = \log |x + 2 + \sqrt{(x^2 + 4x + 13)}|$.

$I = 3\sqrt{(x^2 + 4x + 13)} - \log |x + 2 + \sqrt{(x^2 + 4x + 13)}| + C$.

OR

(B) Integrating by parts with $\log (x^2 + 4)$ as the first function and 1 as the second:

$I = x \log (x^2 + 4) - \int x \cdot 2x/(x^2 + 4)\, dx$

$= x \log (x^2 + 4) - 2 \int [1 - 4/(x^2 + 4)]\, dx$ (since $x^2 = (x^2 + 4) - 4$)

$= x \log (x^2 + 4) - 2x + 8 \cdot (1/2) \tan^{-1}(x/2) + C$

$I = x \log (x^2 + 4) - 2x + 4 \tan^{-1}(x/2) + C$.

Marking: (A) Splitting $3x + 5 = (3/2)(2x + 4) - 1$ – 1 mark · (A) First integral $3\sqrt{(x^2 + 4x + 13)}$ – 1 mark · (A) Second integral (completing the square) – 1 mark · (B) Integration by parts with 1 as the second function – 1 mark · (B) Writing $2x^2/(x^2 + 4) = 2 - 8/(x^2 + 4)$ – 1 mark · (B) Final answer – 1 mark

Examiner tip: In (A) the coefficient is 3/2, not 3: match 3x with (3/2)(2x). In (B), take 1 as the second function and divide x^2 by $x^2 + 4$ before integrating; $\int dx/(x^2 + 4) = (1/2) \tan^{-1}(x/2)$.

29. (A) Find the particular solution of the differential equation $2xy\,(dy/dx) = x^2 + 3y^2$, $x > 0$, given that $y = 2$ when $x = 1$.

OR

(B) Find the particular solution of the differential equation $x\,(dy/dx) + y = x \cos x$, $x > 0$, given that $y = 1$ when $x = \pi/2$.

Answer (3 marks)

(A) $dy/dx = (x^2 + 3y^2)/(2xy)$, a homogeneous equation (both numerator and denominator are of degree 2). Put $y = vx$, $dy/dx = v + x\, dv/dx$:

$v + x\, dv/dx = (1 + 3v^2)/(2v) \Rightarrow x\, dv/dx = (1 + 3v^2 - 2v^2)/(2v) = (1 + v^2)/(2v)$

$\Rightarrow 2v\, dv/(1 + v^2) = dx/x \Rightarrow \log (1 + v^2) = \log x + \log C \Rightarrow 1 + v^2 = Cx$

$\Rightarrow 1 + y^2/x^2 = Cx \Rightarrow x^2 + y^2 = Cx^3$ (general solution).

$y = 2$ at $x = 1$: $1 + 4 = C \Rightarrow C = 5$.

Particular solution: **$x^2 + y^2 = 5x^3$**.

OR

(B) Dividing by x: $dy/dx + (1/x)\, y = \cos x$, linear with $P = 1/x$, $Q = \cos x$.

I.F. $= e^{\int (1/x)\, dx} = e^{\log x} = x$.

Solution: $y \cdot x = \int x \cos x\, dx = x \sin x + \cos x + C$ (by parts).

$y = 1$ at $x = \pi/2$: $\pi/2 = (\pi/2)(1) + 0 + C \Rightarrow C = 0$.

Particular solution: **$xy = x \sin x + \cos x$**, i.e. $y = \sin x + (\cos x)/x$.

Marking: (A) Substitution $y = vx$ and separating variables – 1 mark · (A) Integration: $x^2 + y^2 = Cx^3$ – 1 mark · (A) $C = 5$ and particular solution – 1 mark · (B) Standard form and I.F. = x – 1 mark · (B) $xy = x \sin x + \cos x + C$ – 1 mark · (B) $C = 0$ and particular solution – 1 mark

Examiner tip: In (A), simplify $(1 + 3v^2)/(2v) - v$ carefully: it is $(1 + v^2)/(2v)$. In (B), divide by x first so that the coefficient of dy/dx is 1 before finding the I.F.

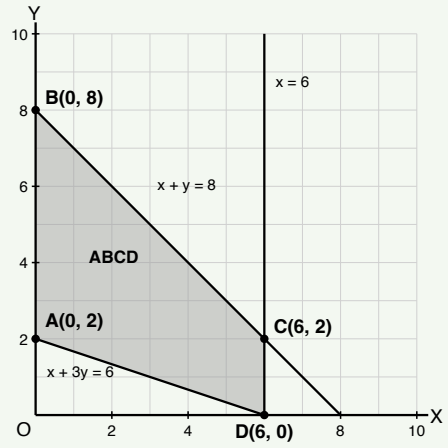
30. Solve the following linear programming problem graphically:
Maximise and minimise Z = 4x + 3y
subject to the constraints $x + y \le 8$, $x + 3y \ge 6$, $x \le 6$, $x \ge 0$, $y \ge 0$.

Answer (3 marks)

Draw the lines x + y = 8 (through (8, 0), (0, 8)), x + 3y = 6 (through (6, 0), (0, 2)) and x = 6.
The feasible region is the bounded quadrilateral ABCD (shaded) with corner points A(0, 2), B(0, 8), C(6, 2) [x = 6 and x + y = 8] and D(6, 0).

Corner point	Z = 4x + 3y
A(0, 2)	6 (minimum)
B(0, 8)	24
C(6, 2)	30 (maximum)
D(6, 0)	24

The region is bounded, so **maximum Z = 30 at x = 6, y = 2** and **minimum Z = 6 at x = 0, y = 2**.



Marking: Correct graph and feasible region – 1 mark · Corner points – 1 mark · Maximum 30 at (6, 2) and minimum 6 at (0, 2) – ½ + ½ mark

Examiner tip: x + 3y ≥ 6 keeps the side AWAY from the origin, so the origin is not feasible and A(0, 2), D(6, 0) are corners, not O(0, 0).

31. (A) Find the coordinates of the foot of the perpendicular drawn from the point P(5, 3, 4) to the line $(x - 1)/2 = y/(-1) = (z - 2)/2$. Hence find the perpendicular distance of P from the line.

OR

(B) Find the shortest distance between the lines $\vec{r} = (\hat{i} - 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} + 2\hat{j} + 2\hat{k})$ and $\vec{r} = (4\hat{i} + \hat{j} + 3\hat{k}) + \mu(2\hat{i} + 4\hat{j} + 4\hat{k})$.

Answer (3 marks)

(A) Any point on the line: $F(1 + 2\lambda, -\lambda, 2 + 2\lambda)$. Direction ratios of the line: 2, −1, 2.
Direction ratios of PF: $2\lambda - 4, -\lambda - 3, 2\lambda - 2$.
PF ⊥ line $\Rightarrow 2(2\lambda - 4) - 1(-\lambda - 3) + 2(2\lambda - 2) = 0 \Rightarrow 9\lambda - 9 = 0 \Rightarrow \lambda = 1$.
Foot of the perpendicular: **F(3, −1, 4)**.
Perpendicular distance $PF = \sqrt{(5 - 3)^2 + (3 + 1)^2 + (4 - 4)^2} = \sqrt{20} = \mathbf{2\sqrt{5}}$ **units**.

OR

(B) $2\hat{i} + 4\hat{j} + 4\hat{k} = 2(\hat{i} + 2\hat{j} + 2\hat{k})$, so the lines are parallel with $\vec{b} = \hat{i} + 2\hat{j} + 2\hat{k}$, $|\vec{b}| = 3$.
$\vec{a}_1 = \hat{i} - 2\hat{j} + 3\hat{k}$, $\vec{a}_2 = 4\hat{i} + \hat{j} + 3\hat{k}$, $\vec{a}_2 - \vec{a}_1 = 3\hat{i} + 3\hat{j}$.

$\vec{b} \times (\vec{a}_2 - \vec{a}_1) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 2 \\ 3 & 3 & 0 \end{vmatrix} = \hat{i}(0-6) - \hat{j}(0-6) + \hat{k}(3-6) = -6\hat{i} + 6\hat{j} - 3\hat{k}$, with magnitude $\sqrt{(36+36+9)} = 9$.

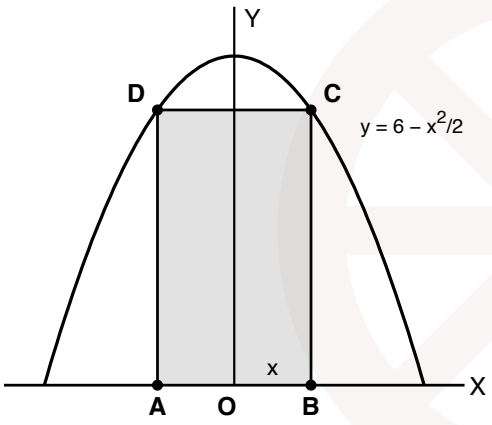
$d = |\vec{b} \times (\vec{a}_2 - \vec{a}_1)| / |\vec{b}| = 9/3 =$ **3 units**.

Marking: (A) General point and direction ratios of PF – 1 mark · (A) $\lambda = 1$ and foot F(3, −1, 4) – 1 mark · (A) Distance $2\sqrt{5}$ – 1 mark · (B) Recognising the lines are parallel and using $d = |\vec{b} \times (\vec{a}_2 - \vec{a}_1)| / |\vec{b}|$ – 1 mark · (B) Cross product – 1 mark · (B) d = 3 units – 1 mark

Examiner tip: In (B) the direction vectors are multiples of each other, so the skew-lines formula (with $\vec{b}_1 \times \vec{b}_2$) cannot be used: it gives 0/0. Use the formula for parallel lines.

SECTION D This section comprises 4 long answer (LA) type questions of 5 marks each. **[20]**

32. The entrance of a school auditorium is a parabolic arch. Taking the mid-point O of the floor line of the arch as the origin, the x-axis along the floor and distances in metres, the arch is the curve $y = 6 - x^2/2$, $y \geq 0$ (see figure). A rectangular glass door ABCD is to be fitted in the arch with its lower edge AB on the floor and its upper corners C and D on the arch. Taking OB = x m, find the dimensions of the door of the largest area and this largest area. Also, using integration, find the area of the opening of the arch and show that the area of the largest door and the area of the opening are in the ratio $1 : \sqrt{3}$.



Answer (5 marks)

The arch meets the floor where $6 - x^2/2 = 0$, i.e. at $x = \pm 2\sqrt{3}$.

By symmetry, A(−x, 0), B(x, 0), C(x, $6 - x^2/2$) and D(−x, $6 - x^2/2$), where $0 < x < 2\sqrt{3}$.

Width AB = 2x and height BC = $6 - x^2/2$.

Area of the door $S(x) = 2x(6 - x^2/2) = 12x - x^3$.

$S'(x) = 12 - 3x^2$; $S'(x) = 0 \Rightarrow x^2 = 4 \Rightarrow x = 2$ ($x > 0$).

$S''(x) = -6x$, so $S''(2) = -12 < 0$. Hence S is maximum at x = 2.

Dimensions of the largest door: width 2x = **4 m** and height $6 - 2 =$ **4 m**; largest area $S(2) = 24 - 8 =$ **16 m²**.

Area of the opening $= \int_{-2\sqrt{3}}^{2\sqrt{3}} (6 - x^2/2)\,dx = 2\int_{0}^{2\sqrt{3}} (6 - x^2/2)\,dx$ (the curve is symmetric about the y-axis)

$= 2[6x - x^3/6]_0^{2\sqrt{3}} = 2(12\sqrt{3} - 24\sqrt{3}/6) = 2(12\sqrt{3} - 4\sqrt{3}) =$ **$16\sqrt{3}$ m²**.

Ratio $= 16 : 16\sqrt{3} =$ **$1 : \sqrt{3}$**. Hence shown.

Marking: Area $S(x) = 2x(6 - x^2/2) = 12x - x^3$ – 1 mark · $S'(x) = 0 \Rightarrow x = 2$ – 1 mark · $S''(2) < 0$, so maximum – 1 mark · Door 4 m × 4 m, largest area 16 m² – 1 mark · Area of the opening $16\sqrt{3}$ m² and ratio $1 : \sqrt{3}$ – 1 mark

Examiner tip: The width of the door is 2x, not x: A and B are each x m from O. For the area of the opening, first find where the arch meets the floor ($x = \pm 2\sqrt{3}$); these are the limits of integration.

33. (A) Evaluate: $\int_0^{\pi/2} \sin^2 x/(2 + \sin 2x)\, dx$

OR

(B) Find: $\int (5x^2 - 4x + 7)/((x - 1)(x^2 + 3))\, dx$

Answer (5 marks)

(A) Let $I = \int_0^{\pi/2} \sin^2 x/(2 + \sin 2x)\, dx$ ... (1)

Using $\int_0^a f(x)\, dx = \int_0^a f(a - x)\, dx$ with $a = \pi/2$, where $\sin(\pi/2 - x) = \cos x$ and $\sin 2(\pi/2 - x) = \sin(\pi - 2x) = \sin 2x$:

$I = \int_0^{\pi/2} \cos^2 x/(2 + \sin 2x)\, dx$ ... (2)

Adding (1) and (2), since $\sin^2 x + \cos^2 x = 1$: $2I = \int_0^{\pi/2} dx/(2 + \sin 2x)$.

Let $g(x) = 1/(2 + \sin 2x)$. As shown above, $g(\pi/2 - x) = g(x)$, so by $\int_0^{2a} g(x)\, dx = 2\int_0^a g(x)\, dx$ (here $2a = \pi/2$):

$2I = 2\int_0^{\pi/4} dx/(2 + \sin 2x) \Rightarrow I = \int_0^{\pi/4} dx/(2 + 2\sin x \cos x)$.

Dividing the numerator and the denominator by $\cos^2 x$: $I = \int_0^{\pi/4} \sec^2 x/(2\sec^2 x + 2\tan x)\, dx = (1/2)\int_0^{\pi/4} \sec^2 x/(\tan^2 x + \tan x + 1)\, dx$.

Put $\tan x = t$, so $\sec^2 x\, dx = dt$; when $x = 0$, $t = 0$ and when $x = \pi/4$, $t = 1$:

$I = (1/2)\int_0^1 dt/((t + 1/2)^2 + (\sqrt{3}/2)^2) = (1/2) \cdot (2/\sqrt{3})\,[\tan^{-1}((2t + 1)/\sqrt{3})]_0^1$

$= (1/\sqrt{3})\,[\tan^{-1}\sqrt{3} - \tan^{-1}(1/\sqrt{3})] = (1/\sqrt{3})(\pi/3 - \pi/6) = (1/\sqrt{3})(\pi/6)$.

Hence $\mathbf{I = \pi/(6\sqrt{3})}$ $(= \sqrt{3}\,\pi/18)$.

OR

(B) Partial fractions: $(5x^2 - 4x + 7)/((x - 1)(x^2 + 3)) = A/(x - 1) + (Bx + C)/(x^2 + 3)$

$\Rightarrow 5x^2 - 4x + 7 = A(x^2 + 3) + (Bx + C)(x - 1)$.

$x = 1$: $8 = 4A \Rightarrow A = 2$.

Coefficient of x^2: $5 = A + B \Rightarrow B = 3$. Constant term: $7 = 3A - C \Rightarrow C = -1$.

So the integrand $= 2/(x - 1) + (3x - 1)/(x^2 + 3) = 2/(x - 1) + (3/2) \cdot 2x/(x^2 + 3) - 1/(x^2 + 3)$.

$\mathbf{I = 2\log|x - 1| + (3/2)\log(x^2 + 3) - (1/\sqrt{3})\tan^{-1}(x/\sqrt{3}) + C}$.

Marking: (A) Using $\int_0^a f(x)\, dx = \int_0^a f(a - x)\, dx$ to get (2) – 1 mark · (A) Adding: $2I = \int_0^{\pi/2} dx/(2 + \sin 2x)$ – 1 mark · (A) Halving the range by $\int_0^{2a} g(x)\, dx = 2\int_0^a g(x)\, dx$ and dividing by $\cos^2 x$ – 1 mark · (A) Substitution $\tan x = t$ with limits 0 and 1, completing the square – 1 mark · (A) $I = \pi/(6\sqrt{3})$ – 1 mark · (B) Correct form of partial fractions – 1 mark · (B) A = 2, B = 3, C = −1 – 2 marks · (B) Splitting $(3x - 1)/(x^2 + 3)$ – 1 mark · (B) Final answer – 1 mark

Examiner tip: In (A), the property turns $\sin^2 x$ into $\cos^2 x$, and adding the two forms clears the numerator. Halve the range ($2 + \sin 2x$ does not change when x is replaced by $\pi/2 - x$) BEFORE putting $\tan x = t$: $\tan x$ is not defined at $x = \pi/2$. In (B), a quadratic factor $x^2 + 3$ needs a LINEAR numerator $Bx + C$, not just a constant.

34. (A) Using integration, find the area of the region bounded by the parabola $y^2 = 4x$ and the line $x - y = 3$.

OR

(B) Using integration, find the area of the triangle whose vertices are A(1, 2), B(3, 6) and C(5, 4).

Answer (5 marks)

(A) Points of intersection: $x = y + 3 \Rightarrow y^2 = 4(y + 3) \Rightarrow y^2 - 4y - 12 = 0 \Rightarrow (y - 6)(y + 2) = 0 \Rightarrow y = 6$ or $y = -2$.

The points are Q(1, −2) and P(9, 6) (see figure).

It is easier to integrate with respect to y: between $y = -2$ and $y = 6$, the line $x = y + 3$ lies to the right of the parabola $x = y^2/4$.

Area $= \int_{-2}^{6} [(y + 3) - y^2/4]\, dy = [y^2/2 + 3y - y^3/12]_{-2}^{6}$

$= (18 + 18 - 18) - (2 - 6 + 2/3) = 18 + 10/3 =$ **64/3 square units**.

OR

(B) Equations of the sides:

AB: $y = 2x$; BC: $y = 9 - x$; AC: $y = (x + 3)/2$.

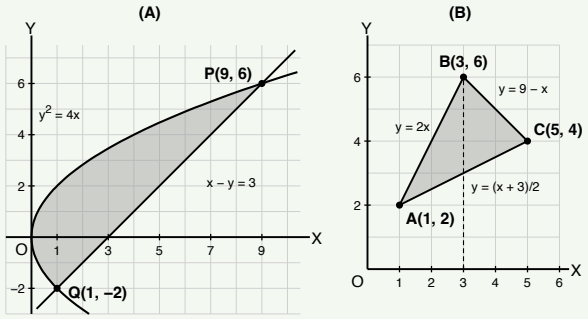
For $1 \le x \le 3$ the region lies between AB (above) and AC (below); for $3 \le x \le 5$ it lies between BC (above) and AC (below).

Area $= \int_{1}^{3} [2x - (x + 3)/2]\, dx + \int_{3}^{5} [(9 - x) - (x + 3)/2]\, dx$

$= \int_{1}^{3} (3x - 3)/2\, dx + \int_{3}^{5} (15 - 3x)/2\, dx$

$= [3x^2/4 - 3x/2]_{1}^{3} + [15x/2 - 3x^2/4]_{3}^{5}$

$= (9/4 + 3/4) + (75/4 - 63/4) = 3 + 3 =$ **6 square units**.



Marking: (A) Points of intersection (1, −2) and (9, 6) – 1 mark · (A) Correct sketch – 1 mark · (A) Correct integral set-up – 1 mark · (A) Integration and limits – 1 mark · (A) Area 64/3 sq. units – 1 mark · (B) Equations of the three sides – 1½ marks · (B) Sketch and set-up of the two integrals – 1½ marks · (B) Evaluation – 1 mark · (B) Area 6 sq. units – 1 mark

Examiner tip: In (A), integrating with respect to x needs two pieces (for $0 \le x \le 1$ the region lies between the two branches $y = \pm 2\sqrt{x}$, for $1 \le x \le 9$ between the line and $y = 2\sqrt{x}$); integrating with respect to y needs just one. In (B), the lower side AC runs the whole way from $x = 1$ to $x = 5$, but the upper boundary changes at $x = 3$.

35. During the monsoon in Mumbai, the probability that it rains on a given day is 0.6. On a rainy day, the probability that a particular suburban local train runs late is 0.5; on a day without rain this probability is 0.1. On a day chosen at random,

(i) find the probability that the train runs late; [2]

(ii) if the train runs late, find the probability that it rained that day; [2]

(iii) if the train is on time, find the probability that it did not rain that day. [1]

Answer (5 marks)

Let E_1: it rains, E_2: it does not rain, A: the train runs late.

$P(E_1) = 0.6$, $P(E_2) = 0.4$, $P(A \mid E_1) = 0.5$, $P(A \mid E_2) = 0.1$.

(i) By the theorem of total probability:

$P(A) = P(E_1)\, P(A \mid E_1) + P(E_2)\, P(A \mid E_2) = 0.6 \times 0.5 + 0.4 \times 0.1 = 0.30 + 0.04 =$ **0.34**.

(ii) By Bayes' theorem:

$P(E_1 \mid A) = P(E_1)\, P(A \mid E_1)/P(A) = 0.30/0.34 =$ **15/17**.

(iii) Let A′: the train is on time. $P(A' \mid E_2) = 1 - 0.1 = 0.9$ and $P(A') = 1 - 0.34 = 0.66$.

$P(E_2 \mid A') = P(E_2)\, P(A' \mid E_2)/P(A') = (0.4 \times 0.9)/0.66 = 0.36/0.66 =$ **6/11**.

Marking: Defining events and writing the probabilities – 1 mark · (i) P(A) = 0.34 – 1 mark · (ii) Bayes' theorem set-up – 1 mark, 15/17 – 1 mark · (iii) 6/11 – 1 mark

Examiner tip: In (iii) use P(on time | no rain) = 0.9 and P(on time) = 0.66; using 0.1 or 0.34 here is the usual slip.

SECTION E This section comprises 3 case study based questions of 4 marks each. Questions 36 and 37 have three sub-parts of 1, 1 and 2 marks; Question 38 has two sub-parts of 2 marks each. **[12]**

36. **Case study:** In a 3-D design model of a cable-stayed footbridge over a river in Nagpur, three straight steel cables are represented by lines. Taking a fixed point on the bank as the origin and distances in metres, the first two cables lie along the lines

$l_1 : (x - 1)/2 = (y + 1)/3 = (z - 2)/6$ and $l_2 : (x + 3)/6 = y/2 = (z - 11)/(-3)$.

Based on the given information, answer the following questions:

(i) Find the direction cosines of the line l_1. [1]

(ii) Show that the lines l_1 and l_2 are perpendicular to each other. [1]

(iii) (a) Show that the lines l_1 and l_2 intersect and find their point of intersection. [2]

OR

(iii) (b) A third cable l_3 passes through the point Q(2, 3, 2) and is parallel to l_2. Find the shortest distance between the lines l_1 and l_3. [2]

Answer (4 marks)

(i) Direction ratios of l_1: 2, 3, 6; $\sqrt{(4 + 9 + 36)} = 7$. Direction cosines: **2/7, 3/7, 6/7**.

(ii) Direction ratios of l_2: 6, 2, −3. $a_1a_2 + b_1b_2 + c_1c_2 = 12 + 6 - 18 = 0$, so $\mathbf{l_1 \perp l_2}$.

(iii) (a) General point of l_1: $(2\lambda + 1, 3\lambda - 1, 6\lambda + 2)$; of l_2: $(6\mu - 3, 2\mu, -3\mu + 11)$.

Equating: $2\lambda + 1 = 6\mu - 3$ … (1); $3\lambda - 1 = 2\mu$ … (2); $6\lambda + 2 = -3\mu + 11$ … (3).

From (2), $\mu = (3\lambda - 1)/2$; in (1): $2\lambda + 4 = 9\lambda - 3 \Rightarrow \lambda = 1, \mu = 1$.

Check in (3): LHS = 8, RHS = −3 + 11 = 8 ✓. So the lines intersect.

Point of intersection: **(3, 2, 8)**.

OR

(iii) (b) $l_3 : \vec{r} = (2\hat{i} + 3\hat{j} + 2\hat{k}) + \mu(6\hat{i} + 2\hat{j} - 3\hat{k})$; $l_1 : \vec{r} = (\hat{i} - \hat{j} + 2\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$.

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 6 \\ 6 & 2 & -3 \end{vmatrix} = \hat{i}(-9 - 12) - \hat{j}(-6 - 36) + \hat{k}(4 - 18) = -21\hat{i} + 42\hat{j} - 14\hat{k},$$ with magnitude $7\sqrt{(9 + 36 + 4)} = 49$.

$\vec{a}_2 - \vec{a}_1 = \hat{i} + 4\hat{j}$; $(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2) = -21 + 168 + 0 = 147$.

Shortest distance = $|147|/49 = \mathbf{3\ m}$.

Marking: (i) Direction cosines – 1 mark · (ii) Dot product of direction ratios = 0 – 1 mark · (iii)(a) $\lambda = 1$, $\mu = 1$ and checking the third equation – 1 mark, point (3, 2, 8) – 1 mark · (iii)(b) $\vec{b}_1 \times \vec{b}_2$ – 1 mark, shortest distance 3 m – 1 mark

Examiner tip: Two lines in space intersect only if the values of λ and μ found from two equations ALSO satisfy the third; always write this check.

37. **Case study:** At a computer centre, a student who has never typed before starts practising touch-typing. Her instructor models her typing speed y (in words per minute) after t weeks of practice as follows: the speed increases at a rate proportional to the difference between a limiting speed of 80 words per minute and her present speed, i.e. $dy/dt = k(80 - y)$, where k is a positive constant, and $y = 0$ when $t = 0$.

Based on the given information, answer the following questions:

(i) Write the order and the degree of the differential equation $dy/dt = k(80 - y)$. [1]

(ii) Solve the differential equation to show that $y = 80(1 - e^{-kt})$. [1]

(iii) (a) If her speed is 40 words per minute after 2 weeks, after how many weeks will her speed be 60 words per minute? [2]

OR

(iii) (b) If her speed is 20 words per minute after 3 weeks, find her speed after 6 weeks. [2]

Answer (4 marks)

(i) The highest derivative is dy/dt (first order) and it occurs with power 1: **order 1, degree 1**.

(ii) Separating the variables: $dy/(80 - y) = k\,dt \Rightarrow -\log(80 - y) = kt + C$ $(y < 80)$.

At $t = 0$, $y = 0 \Rightarrow C = -\log 80$. So $\log(80/(80 - y)) = kt \Rightarrow 80 - y = 80\,e^{-kt} \Rightarrow$ **$y = 80(1 - e^{-kt})$**.

(iii) (a) $y = 40$ at $t = 2 \Rightarrow 1 - e^{-2k} = 1/2 \Rightarrow e^{-2k} = 1/2$.

$y = 60 \Rightarrow 1 - e^{-kt} = 3/4 \Rightarrow e^{-kt} = 1/4 = (1/2)^2 = (e^{-2k})^2 = e^{-4k} \Rightarrow t = 4$.

Her speed will be 60 words per minute after **4 weeks**.

OR

(iii) (b) $y = 20$ at $t = 3 \Rightarrow 1 - e^{-3k} = 1/4 \Rightarrow e^{-3k} = 3/4$.

At $t = 6$: $y = 80(1 - e^{-6k}) = 80(1 - (3/4)^2) = 80 \times 7/16 =$ **35 words per minute**.

Marking: (i) Order 1 and degree 1 – ½ + ½ mark · (ii) Separating the variables, integrating and using $y(0) = 0$ – 1 mark · (iii)(a) $e^{-2k} = 1/2$ – 1 mark, $t = 4$ weeks – 1 mark · (iii)(b) $e^{-3k} = 3/4$ – 1 mark, 35 words per minute – 1 mark

Examiner tip: $\int dy/(80 - y) = -\log(80 - y)$: do not lose the minus sign. No calculator is needed: work with $e^{-2k} = 1/2$ (or $e^{-3k} = 3/4$) as a block instead of finding k as a decimal.

38. **Case study:** Three farmers of a co-operative society in Satara buy bags of three fertilisers: urea, DAP and potash. Farmer 1 buys 2 bags of urea, 1 bag of DAP and 1 bag of potash for ₹3,650. Farmer 2 buys 1 bag of urea, 2 bags of DAP and 1 bag of potash for ₹4,700. Farmer 3 buys 3 bags of urea, 1 bag of DAP and 2 bags of potash for ₹5,650. Let the price of one bag of urea, DAP and potash be ₹x, ₹y and ₹z respectively.

Based on the given information, answer the following questions:

(i) Write the above information as a matrix equation AX = B, and find $|A|$ and adj A. [2]

(ii) Find A^{-1} and hence find the price of one bag of each fertiliser. [2]

Answer (4 marks)

(i) $2x + y + z = 3650$, $x + 2y + z = 4700$, $3x + y + 2z = 5650$.

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 3 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3650 \\ 4700 \\ 5650 \end{bmatrix}, \text{ i.e. } AX = B.$$

$|A| = 2(4 - 1) - 1(2 - 3) + 1(1 - 6) = 6 + 1 - 5 = \mathbf{2} \neq 0$.

Cofactors: $A_{11} = 3$, $A_{12} = -(2 - 3) = 1$, $A_{13} = 1 - 6 = -5$;

$A_{21} = -(2 - 1) = -1$, $A_{22} = 4 - 3 = 1$, $A_{23} = -(2 - 3) = 1$;

$A_{31} = 1 - 2 = -1$, $A_{32} = -(2 - 1) = -1$, $A_{33} = 4 - 1 = 3$.

$$\text{adj } A = \begin{bmatrix} 3 & -1 & -1 \\ 1 & 1 & -1 \\ -5 & 1 & 3 \end{bmatrix}$$

(ii) $A^{-1} = (1/|A|)\ \text{adj}\ A = (1/2)\begin{bmatrix} 3 & -1 & -1 \\ 1 & 1 & -1 \\ -5 & 1 & 3 \end{bmatrix}$

$X = A^{-1}B = (1/2)\begin{bmatrix} 10950 - 4700 - 5650 \\ 3650 + 4700 - 5650 \\ -18250 + 4700 + 16950 \end{bmatrix} = (1/2)\begin{bmatrix} 600 \\ 2700 \\ 3400 \end{bmatrix} = \begin{bmatrix} 300 \\ 1350 \\ 1700 \end{bmatrix}$

Price of one bag: **urea ₹300, DAP ₹1,350, potash ₹1,700**.

Check: 600 + 1350 + 1700 = 3650 ✓; 300 + 2700 + 1700 = 4700 ✓; 900 + 1350 + 3400 = 5650 ✓.

Marking: (i) Matrix equation – ½ mark, $|A| = 2$ – ½ mark, adj A – 1 mark · (ii) A^{-1} – ½ mark, $X = A^{-1}B$ – ½ mark, prices – 1 mark

Examiner tip: adj A is the transpose of the matrix of cofactors. Substitute the prices back into all three equations to catch sign slips.

All the best for your exam!