Mathematics (041)

CBSE Class 12 Board Examination 2027 – Practice Paper No. 1

Model Answer Paper

Time: 3 hours **Max. Marks: 80**

Model answers show one correct method; any other correct method gets full marks.

SECTION A This section comprises 18 multiple choice questions (Q1–Q18) of 1 mark each. Choose the correct option. **[18]**

1. Let A = {2, 4, 6} and let R = {(2, 2), (4, 4), (6, 6), (2, 4), (4, 6)} be a relation on A. The minimum number of ordered pairs that must be added to R so that R becomes an equivalence relation is

(A) 1 (B) 2

(C) 3 (D) 4

Answer (1 mark)

(D) 4

R is already reflexive. For symmetry we need (4, 2) and (6, 4). Since (2, 4) ∈ R and (4, 6) ∈ R, transitivity needs (2, 6), and then symmetry needs (6, 2).

Adding (4, 2), (6, 4), (2, 6), (6, 2) gives R = A × A, which is an equivalence relation, and none of these four pairs can be left out. So 4 pairs.

Marking: Correct option – 1 mark

2. The function $f : N \to N$ defined by $f(n) = n^2 + n$ is

(A) one-one but not onto (B) onto but not one-one

(C) both one-one and onto (D) neither one-one nor onto

Answer (1 mark)

(A) one-one but not onto

If $m < n$ ($m, n \in N$), then $m^2 + m < n^2 + n$, so different inputs give different images: f is one-one.

$f(n) = n(n + 1)$ is always even and $f(n) \geq 2$, so 1 (or 3) has no pre-image: f is not onto.

Marking: Correct option – 1 mark

3. The value of $\cot^{-1}(-1) + \operatorname{cosec}^{-1}(-\sqrt{2}) + \sec^{-1}(\sqrt{2})$ is

(A) $\pi/4$ (B) $3\pi/4$

(C) $-\pi/4$ (D) $5\pi/4$

Answer (1 mark)

(B) $3\pi/4$

Principal value branches: $\cot^{-1} \to (0, \pi)$, $\operatorname{cosec}^{-1} \to [-\pi/2, \pi/2] - \{0\}$, $\sec^{-1} \to [0, \pi] - \{\pi/2\}$.

$\cot^{-1}(-1) = 3\pi/4$, $\operatorname{cosec}^{-1}(-\sqrt{2}) = -\pi/4$, $\sec^{-1}(\sqrt{2}) = \pi/4$.

Sum $= 3\pi/4 - \pi/4 + \pi/4 = 3\pi/4$.

Marking: Correct option – 1 mark

Examiner tip: $\cot^{-1}(-1)$ is $3\pi/4$, not $-\pi/4$: the principal value of $\cot^{-1}$ always lies in $(0, \pi)$.

4. If A is a matrix of order 3 × 4 and B is a matrix such that both A′B and BA′ are defined, then the order of B is

(A) 3 × 3 (B) 4 × 4

(C) 3 × 4 (D) 4 × 3

Answer (1 mark)

(C) 3 × 4

A′ is of order 4 × 3. A′B is defined ⇒ B has 3 rows. BA′ is defined ⇒ B has 4 columns. So B is of order 3 × 4.

Marking: Correct option – 1 mark

5. For the matrix $A = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$, it is given that $A^2 - 5A + 6I = O$, where I is the identity matrix of order 2. Using this, A^{-1} is equal to

(A) (1/6)(5I − A) (B) (1/6)(A − 5I)

(C) (1/6)(5I + A) (D) 6(5I − A)

Answer (1 mark)

(A) (1/6)(5I − A)

$|A| = 4 + 2 = 6 \neq 0$, so A^{-1} exists. Multiplying $A^2 - 5A + 6I = O$ by A^{-1}: $A - 5I + 6A^{-1} = O \Rightarrow A^{-1} = (1/6)(5I - A)$.

Check: $(1/6)(5I - A) = (1/6)\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}$, and $A \cdot (1/6)\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix} = (1/6)\begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} = I$.

Marking: Correct option – 1 mark

6. If A is a square matrix of order 3 and $|A| = -4$, then $|\text{adj}\,(2A)|$ is equal to

(A) 256 (B) −1024

(C) 64 (D) 1024

Answer (1 mark)

(D) 1024

For a square matrix B of order n, $|\text{adj}\,B| = |B|^{n-1}$.

$|2A| = 2^3 \times (-4) = -32$, so $|\text{adj}\,(2A)| = (-32)^2 = 1024$.

Marking: Correct option – 1 mark

Examiner tip: $|kA| = k^n|A|$ for a matrix of order n; students often write $|2A| = 2|A|$.

7. If the area of the triangle with vertices (1, 2), (k, −1) and (4, 3) is 6 square units, then the value(s) of k is/are

(A) 4 only (B) −20 only

(C) 4 or −20 (D) −4 or 20

Answer (1 mark)

(C) 4 or −20

$\text{Area} = \frac{1}{2}\left|\begin{vmatrix} 1 & 2 & 1 \\ k & -1 & 1 \\ 4 & 3 & 1 \end{vmatrix}\right| = \frac{1}{2}|1(-1-3) - 2(k-4) + 1(3k+4)| = \frac{1}{2}|k+8|$.

$\frac{1}{2}|k + 8| = 6 \Rightarrow k + 8 = \pm 12 \Rightarrow k = 4$ or $k = -20$.

Marking: Correct option – 1 mark

8. If the function f defined by $f(x) = (1 - \cos 6x)/(3x^2)$ for $x \neq 0$ and $f(0) = k$ is continuous at $x = 0$, then k is equal to

(A) 6 (B) 3

(C) 9 (D) 12

Answer (1 mark)

(A) 6

$1 - \cos 6x = 2 \sin^2 3x$. So $\lim_{x \to 0} f(x) = \lim_{x \to 0} (2 \sin^2 3x)/(3x^2) = (2 \times 9/3) \cdot \lim_{x \to 0} (\sin 3x / 3x)^2 = 6 \times 1 = 6$.

For continuity, $k = f(0) = 6$.

Marking: Correct option – 1 mark

9. The derivative of $\log (\sec x + \tan x)$ with respect to x is

(A) $\tan x$ (B) $\sec x$

(C) $\sec^2 x$ (D) $\sec x \tan x$

Answer (1 mark)

(B) $\sec x$

$d/dx [\log (\sec x + \tan x)] = (\sec x \tan x + \sec^2 x)/(\sec x + \tan x) = \sec x (\tan x + \sec x)/(\sec x + \tan x) = \sec x$.

Marking: Correct option – 1 mark

10. The rate of change of the area of a circle with respect to its circumference, when the radius is 7 cm, is

(A) 14 cm^2/cm (B) 14π cm^2/cm

(C) 7/2 cm^2/cm (D) 7 cm^2/cm

Answer (1 mark)

(D) 7 cm^2/cm

$A = \pi r^2$, $C = 2\pi r$. $dA/dC = (dA/dr)/(dC/dr) = 2\pi r/(2\pi) = r$. At $r = 7$ cm, $dA/dC = 7$ cm^2/cm.

Marking: Correct option – 1 mark

Examiner tip: Differentiate with respect to the circumference, not the radius: $dA/dr = 14\pi$ is a different rate.

11. The minimum value of $f(x) = x^2 + 16/x^2$, $x \neq 0$, is

(A) 8 (B) 4

(C) 16 (D) 0

Answer (1 mark)

(A) 8

$f'(x) = 2x - 32/x^3 = 0 \Rightarrow x^4 = 16 \Rightarrow x = \pm 2$. $f''(x) = 2 + 96/x^4 > 0$, so these are points of local minimum.

$f(\pm 2) = 4 + 4 = 8$. (Also $f(x) \to \infty$ as $x \to 0$ or $|x| \to \infty$, so 8 is the minimum value.)

Marking: Correct option – 1 mark

12. $\int 2^x \cdot 3^x \, dx$ is equal to

(A) $6^x \log 6 + C$ (B) $6^{x+1}/(x + 1) + C$

(C) $6^x/\log 6 + C$ (D) $(2^x \cdot 3^x)/(\log 2 \cdot \log 3) + C$

Answer (1 mark)

(C) $6^x/\log 6 + C$

$2^x \cdot 3^x = 6^x$, and $\int a^x \, dx = a^x/\log a + C$. So the integral is $6^x/\log 6 + C$.

Marking: Correct option – 1 mark

Examiner tip: Do not use the power rule for a^x: the variable is in the exponent.

13. The value of $\int_{-2}^{2} (x^5 + x^3 \cos x + 3x^2)\, dx$ is

(A) 0 (B) 16

(C) 8 (D) 32

Answer (1 mark)

(B) 16

x^5 and $x^3 \cos x$ are odd functions, so their integrals over $[-2, 2]$ are 0. $3x^2$ is even.

Value $= 2 \int_0^2 3x^2\, dx = 2 [x^3]_0^2 = 2 \times 8 = 16$.

Marking: Correct option – 1 mark

14. The sum of the order and the degree of the differential equation $(d^3y/dx^3)^2 - 4(dy/dx)^5 + y = 0$ is

(A) 6 (B) 8

(C) 7 (D) 5

Answer (1 mark)

(D) 5

Highest order derivative is d^3y/dx^3, so order = 3. The equation is a polynomial in the derivatives and the highest power of d^3y/dx^3 is 2, so degree = 2. Sum = 5.

Marking: Correct option – 1 mark

Examiner tip: Degree is the power of the HIGHEST order derivative, not the highest power appearing anywhere (here 5).

15. The integrating factor of the differential equation $x\,(dy/dx) + 3y = e^x$, $x > 0$, is

(A) $1/x^3$ (B) x^3

(C) $3 \log x$ (D) e^{3x}

Answer (1 mark)

(B) x^3

Dividing by x: $dy/dx + (3/x)\, y = e^x/x$, so $P = 3/x$.

I.F. $= e^{\int P\, dx} = e^{3 \log x} = x^3$.

Marking: Correct option – 1 mark

16. The projection of the vector $2\hat{i} + 3\hat{j} - \hat{k}$ on the vector $\hat{i} - 2\hat{j} + 2\hat{k}$ is

(A) 2 (B) −6

(C) $-6/\sqrt{14}$ (D) −2

Answer (1 mark)

(D) −2

Projection of $\vec{a}$ on $\vec{b} = (\vec{a} \cdot \vec{b})/|\vec{b}| = (2 - 6 - 2)/\sqrt{(1 + 4 + 4)} = -6/3 = -2$.

Marking: Correct option – 1 mark

17. The direction cosines of the line directed from P(1, −2, 3) to Q(3, 4, 6) are

(A) 2/7, 6/7, 3/7 (B) 2, 6, 3

(C) −2/7, −6/7, −3/7 (D) 2/11, 6/11, 3/11

Answer (1 mark)

(A) 2/7, 6/7, 3/7

Direction ratios of PQ: 3 − 1, 4 − (−2), 6 − 3 = 2, 6, 3. PQ $= \sqrt{(4 + 36 + 9)} = 7$.

Direction cosines (from P to Q): 2/7, 6/7, 3/7.

Marking: Correct option – 1 mark

18. If $P(A) = 0.5$, $P(B) = 0.4$ and $P(A \cup B) = 0.7$, then $P(A \mid A \cup B)$ is equal to

(A) 2/7 (B) 1/2

(C) 5/7 (D) 4/7

Answer (1 mark)

(C) 5/7

$A \cap (A \cup B) = A$. So $P(A \mid A \cup B) = P(A)/P(A \cup B) = 0.5/0.7 = 5/7$.

Marking: Correct option – 1 mark

SECTION A (contd.) Assertion–Reason based questions (Q19–Q20), 1 mark each. Two statements are given, one labelled Assertion (A) and the other labelled Reason (R). Select the correct answer from the options (A), (B), (C) and (D) given below each question. **[2]**

19. **Assertion (A):** The matrix $M = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ is not invertible.

Reason (R): A square matrix is invertible if and only if it is non-singular.

(A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of Assertion (A).

(B) Both Assertion (A) and Reason (R) are true but Reason (R) is not the correct explanation of Assertion (A).

(C) Assertion (A) is true but Reason (R) is false.

(D) Assertion (A) is false but Reason (R) is true.

Answer (1 mark)

(A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of Assertion (A).

$|M| = 4 - 4 = 0$, so M is singular. By (R), a singular matrix is not invertible, so (A) is true and (R), which is a true statement, explains it.

Marking: Correct option – 1 mark

20. **Assertion (A):** The angle between the vectors $\hat{i} + \hat{j}$ and $\hat{j} + \hat{k}$ is $\pi/2$.

Reason (R): Two non-zero vectors are perpendicular if and only if their scalar product is zero.

(A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of Assertion (A).

(B) Both Assertion (A) and Reason (R) are true but Reason (R) is not the correct explanation of Assertion (A).

(C) Assertion (A) is true but Reason (R) is false.

(D) Assertion (A) is false but Reason (R) is true.

Answer (1 mark)

(D) Assertion (A) is false but Reason (R) is true.

$(\hat{i} + \hat{j}) \cdot (\hat{j} + \hat{k}) = 0 + 1 + 0 = 1 \neq 0$. $\cos\theta = 1/(\sqrt{2} \cdot \sqrt{2}) = 1/2$, so $\theta = \pi/3$, not $\pi/2$. (A) is false.

(R) is a true property of the scalar product.

Marking: Correct option – 1 mark

SECTION B This section comprises 5 very short answer (VSA) type questions of 2 marks each. **[10]**

21. (A) Find the value of k for which the function f defined by $f(x) = 3x - 2$ for $x \leq 2$ and $f(x) = x^2 - k$ for $x > 2$ is continuous at $x = 2$. For this value of k, check whether f is differentiable at $x = 2$.

OR

(B) If $y = x \sin^{-1} x + \sqrt{(1 - x^2)}$, $-1 < x < 1$, find dy/dx and hence find its value at $x = 1/2$.

Answer (2 marks)

(A) $f(2) = 3(2) - 2 = 4$. LHL $= \lim_{x \to 2^-} (3x - 2) = 4$. RHL $= \lim_{x \to 2^+} (x^2 - k) = 4 - k$.

Continuity at $x = 2 \Rightarrow 4 - k = 4 \Rightarrow$ **k = 0**.

With k = 0:

$Lf'(2) = \lim_{h\to 0} [f(2-h) - f(2)]/(-h) = \lim_{h\to 0} [3(2-h) - 2 - 4]/(-h) = \lim_{h\to 0} (-3h)/(-h) = 3.$

$Rf'(2) = \lim_{h\to 0} [f(2+h) - f(2)]/h = \lim_{h\to 0} [(2+h)^2 - 4]/h = \lim_{h\to 0} (4+h) = 4.$

$Lf'(2) \neq Rf'(2)$, so **f is not differentiable at x = 2** (though it is continuous there).

OR

(B) $dy/dx = \sin^{-1} x + x \cdot 1/\sqrt{(1-x^2)} + (-2x)/(2\sqrt{(1-x^2)})$

$= \sin^{-1} x + x/\sqrt{(1-x^2)} - x/\sqrt{(1-x^2)} = \mathbf{\sin^{-1} x}.$

At $x = 1/2$: $dy/dx = \sin^{-1}(1/2) = \mathbf{\pi/6}$.

Marking: (A) Value k = 0 – 1 mark · LHD = 3, RHD = 4 and conclusion – 1 mark · (B) Correct differentiation (product rule + chain rule) – 1 mark · Simplification to $\sin^{-1} x$ and value π/6 – 1 mark

Examiner tip: In (A), continuity does not imply differentiability: always find both one-sided derivatives. In (B), the middle terms cancel – check signs of the chain rule.

22. Find the intervals in which the function $f(x) = x^4 - 4x^3 - 2x^2 + 12x + 7$ is (i) strictly increasing, (ii) strictly decreasing.

Answer (2 marks)

$f'(x) = 4x^3 - 12x^2 - 4x + 12 = 4(x^3 - 3x^2 - x + 3) = 4[x^2(x-3) - (x-3)] = 4(x-3)(x-1)(x+1).$

$f'(x) = 0 \Rightarrow x = -1, 1, 3$. These points divide the real line into four intervals:

Interval	Sign of f′(x)	Nature of f
$(-\infty, -1)$	(−)(−)(−) = −	strictly decreasing
$(-1, 1)$	(−)(−)(+) = +	strictly increasing
$(1, 3)$	(−)(+)(+) = −	strictly decreasing
$(3, \infty)$	(+)(+)(+) = +	strictly increasing

(i) f is strictly increasing in **$(-1, 1)$ and $(3, \infty)$**.

(ii) f is strictly decreasing in **$(-\infty, -1)$ and $(1, 3)$**.

Marking: f′(x) = 4(x − 3)(x − 1)(x + 1) – 1 mark · Correct intervals of increase and decrease – ½ + ½ mark

Examiner tip: Factorise f′(x) completely and check the sign in each interval; the signs in the table are for (x − 3)(x − 1)(x + 1) in that order.

23. (A) The diagonals of a parallelogram are given by the vectors $\vec{d}_1 = 3\hat{i} + \hat{j} - 2\hat{k}$ and $\vec{d}_2 = \hat{i} - 3\hat{j} + 4\hat{k}$. Find the area of the parallelogram.

OR

(B) The cartesian equation of a line is $(2x-1)/4 = (y+3)/(-2) = 3 - z$. Find the direction cosines of the line and write its vector equation.

Answer (2 marks)

(A) Area of a parallelogram with diagonals $\vec{d}_1, \vec{d}_2 = \tfrac{1}{2} |\vec{d}_1 \times \vec{d}_2|$.

$$\vec{d}_1 \times \vec{d}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & -2 \\ 1 & -3 & 4 \end{vmatrix} = \hat{i}(4-6) - \hat{j}(12+2) + \hat{k}(-9-1) = -2\hat{i} - 14\hat{j} - 10\hat{k}.$$

$|\vec{d}_1 \times \vec{d}_2| = \sqrt{(4 + 196 + 100)} = \sqrt{300} = 10\sqrt{3}$.

Area = $\tfrac{1}{2} \times 10\sqrt{3} = \mathbf{5\sqrt{3}}$ **square units**.

OR

(B) Write the line in standard form: $(x - 1/2)/2 = (y+3)/(-2) = (z-3)/(-1)$.

It passes through $(1/2, -3, 3)$ and has direction ratios $2, -2, -1$; $\sqrt{(4+4+1)} = 3$.

Direction cosines: **2/3, −2/3, −1/3**.

Vector equation: $\vec{r} = (½\hat{i} - 3\hat{j} + 3\hat{k}) + \lambda(2\hat{i} - 2\hat{j} - \hat{k})$, $\lambda \in R$.

Marking: (A) Cross product of diagonals – 1 mark · Area = $½|\vec{d}_1 \times \vec{d}_2| = 5\sqrt{3}$ sq. units – 1 mark · (B) Standard form and direction cosines – 1 mark · Vector equation – 1 mark

Examiner tip: (A) With diagonals the area is HALF the magnitude of the cross product. (B) Make the coefficients of x, y, z equal to 1 first: 3 − z = (z − 3)/(−1) gives direction ratio −1, not 1.

24. Find: $\int \sec^2 x / (\tan^2 x + 4 \tan x + 8)\, dx$

Answer (2 marks)

Put $\tan x = t \Rightarrow \sec^2 x\, dx = dt$.

$I = \int dt/(t^2 + 4t + 8) = \int dt/((t + 2)^2 + 2^2)$

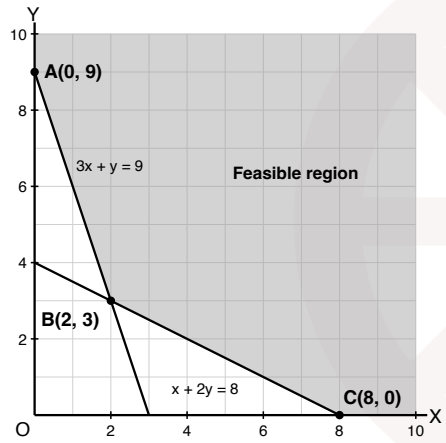
$= ½ \tan^{-1}((t + 2)/2) + C$

$= \mathbf{½ \tan^{-1}((\tan x + 2)/2) + C}$.

Marking: Substitution $\tan x = t$ – ½ mark · Completing the square – ½ mark · Standard integral and final answer – 1 mark

Examiner tip: Use $\int dx/(x^2 + a^2) = (1/a)\tan^{-1}(x/a) + C$; here a = 2, so the factor is ½, not 2.

25. The shaded region in the figure is the feasible region of a linear programming problem with constraints $x + 2y \geq 8$, $3x + y \geq 9$, $x \geq 0$, $y \geq 0$. Its corner points are A(0, 9), B(2, 3) and C(8, 0). Find the minimum value of Z = 4x + 7y, if it exists. Give a reason for your answer.



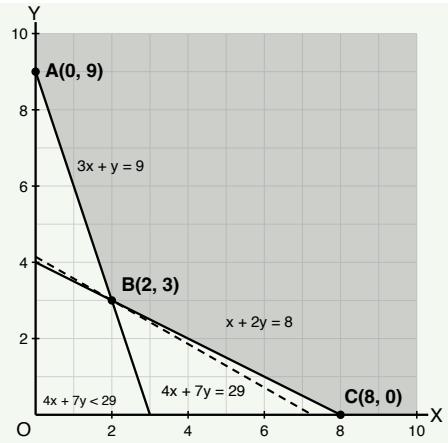
Answer (2 marks)

Corner point	Z = 4x + 7y
A(0, 9)	63
B(2, 3)	29 (smallest)
C(8, 0)	32

The feasible region is unbounded, so 29 is the minimum only if the open half-plane $4x + 7y < 29$ has no point in common with the feasible region.

The line $4x + 7y = 29$ passes through B(2, 3) and (29/4, 0). The half-plane $4x + 7y < 29$ lies on the origin side of it, and it does not meet the feasible region (see figure). [Check: for every feasible point, $4x + 7y = (17/5)(x + 2y) + (1/5)(3x + y) \geq (17/5)(8) + (1/5)(9) = 29$.]

Hence **minimum Z = 29 at B(2, 3)**, i.e. at x = 2, y = 3.



Marking: Values of Z at all corner points – 1 mark · Half-plane check for the unbounded region and minimum Z = 29 at (2, 3) – 1 mark

Examiner tip: For an unbounded feasible region, the smallest corner value is the minimum only after the half-plane check. Students lose the second mark by skipping it.

SECTION C This section comprises 6 short answer (SA) type questions of 3 marks each. **[18]**

26. (A) Find: $\int (x + 7)/((x - 1)(x + 2)(x + 3))\, dx$

OR

(B) Find: $\int e^{3x} \cos 4x\, dx$

Answer (3 marks)

(A) Let $(x + 7)/((x - 1)(x + 2)(x + 3)) = A/(x - 1) + B/(x + 2) + C/(x + 3)$.

Then $x + 7 = A(x + 2)(x + 3) + B(x - 1)(x + 3) + C(x - 1)(x + 2)$.

$x = 1$: $8 = 12A \Rightarrow A = 2/3$. $x = -2$: $5 = -3B \Rightarrow B = -5/3$. $x = -3$: $4 = 4C \Rightarrow C = 1$.

$I = (2/3)\int dx/(x - 1) - (5/3)\int dx/(x + 2) + \int dx/(x + 3)$

$= \mathbf{(2/3) \log |x - 1| - (5/3) \log |x + 2| + \log |x + 3| + C}$.

OR

(B) Let $I = \int e^{3x} \cos 4x\, dx$. Integrating by parts with cos 4x as the first function:

$I = \cos 4x \cdot (e^{3x}/3) - \int (-4 \sin 4x)(e^{3x}/3)\, dx = (1/3)\, e^{3x} \cos 4x + (4/3) \int e^{3x} \sin 4x\, dx$.

Again by parts: $\int e^{3x} \sin 4x\, dx = (1/3)\, e^{3x} \sin 4x - (4/3) \int e^{3x} \cos 4x\, dx = (1/3)\, e^{3x} \sin 4x - (4/3)\, I$.

So $I = (1/3)\, e^{3x} \cos 4x + (4/9)\, e^{3x} \sin 4x - (16/9)\, I$

$\Rightarrow (25/9)\, I = (e^{3x}/9)(3 \cos 4x + 4 \sin 4x)$

$\Rightarrow \mathbf{I = (e^{3x}/25)(3 \cos 4x + 4 \sin 4x) + C}$.

Marking: (A) Partial fraction form – ½ mark · Values A = 2/3, B = −5/3, C = 1 – 1½ marks · Integration – 1 mark · (B) First integration by parts – 1 mark · Second integration by parts – 1 mark · Solving for I – 1 mark

Examiner tip: (A) Check: A + B + C must be 0 because the numerator has lower degree – a quick test of your constants. (B) Keep the same type of function (trig) as the first function in both steps, otherwise you get back to the start.

27. Evaluate: $\int_0^2 x^2 \sqrt{(4 - x^2)}\, dx$

Answer (3 marks)

Put $x = 2 \sin \theta \Rightarrow dx = 2 \cos \theta\, d\theta$. When $x = 0$, $\theta = 0$; when $x = 2$, $\theta = \pi/2$. $\sqrt{(4 - x^2)} = 2 \cos \theta$.

$I = \int_0^{\pi/2} 4 \sin^2 \theta \cdot 2 \cos \theta \cdot 2 \cos \theta\, d\theta = 16 \int_0^{\pi/2} \sin^2 \theta \cos^2 \theta\, d\theta$

$= 4 \int_0^{\pi/2} \sin^2 2\theta\, d\theta = 2 \int_0^{\pi/2} (1 - \cos 4\theta)\, d\theta$

$= 2\, [\theta - (\sin 4\theta)/4]_0^{\pi/2} = 2\, [(\pi/2 - 0) - 0] = \boldsymbol{\pi}$.

Marking: Substitution with new limits – 1 mark · Reducing to 2(1 − cos 4θ) – 1 mark · Evaluation = π – 1 mark

Examiner tip: Change the limits when you substitute; then you need not go back to x.

28. (A) Find the particular solution of the differential equation $(x + 1)(dy/dx) - 2y = (x + 1)^4$, $x > -1$, given that $y = 1$ when $x = 0$.

OR

(B) Find the particular solution of the differential equation $x\,(dy/dx) = y + 2x\,e^{-y/x}$, $x > 0$, given that $y = 0$ when $x = 1$.

Answer (3 marks)

(A) Dividing by $(x + 1)$: $dy/dx - (2/(x + 1))\,y = (x + 1)^3$. This is linear with $P = -2/(x + 1)$, $Q = (x + 1)^3$.

I.F. $= e^{\int P\,dx} = e^{-2\log(x+1)} = 1/(x + 1)^2$.

Solution: $y \cdot 1/(x + 1)^2 = \int (x + 1)^3 \cdot 1/(x + 1)^2\,dx = \int (x + 1)\,dx = (x + 1)^2/2 + C$

$\Rightarrow y = (x + 1)^4/2 + C(x + 1)^2$.

$y = 1$ at $x = 0$: $1 = 1/2 + C \Rightarrow C = 1/2$.

Particular solution: $\mathbf{y = \tfrac{1}{2}[(x + 1)^4 + (x + 1)^2]}$.

OR

(B) $dy/dx = y/x + 2e^{-y/x} = g(y/x)$, so the equation is homogeneous. Put $y = vx \Rightarrow dy/dx = v + x\,dv/dx$.

$v + x\,dv/dx = v + 2e^{-v} \Rightarrow e^v\,dv = 2\,dx/x$.

Integrating: $e^v = 2\log x + C \Rightarrow e^{y/x} = 2\log x + C$.

$y = 0$ at $x = 1$: $e^0 = 0 + C \Rightarrow C = 1$.

Particular solution: $\mathbf{e^{y/x} = 2\log x + 1}$ (or $y = x\log(1 + 2\log x)$).

Marking: (A) Standard linear form and I.F. $= 1/(x + 1)^2$ – 1 mark · Integration and general solution – 1 mark · Value of C and particular solution – 1 mark · (B) Showing homogeneous and substituting $y = vx$ – 1 mark · Separating and integrating – 1 mark · Value of C and particular solution – 1 mark

Examiner tip: Divide by the coefficient of dy/dx before reading off P; in (A) $P = -2/(x + 1)$, so the I.F. is $1/(x + 1)^2$, not $(x + 1)^2$.

29. A shopkeeper in Kolhapur has a box of 8 LED bulbs, of which 3 are defective. A customer takes out three bulbs one after another without replacement. Find the probability that

(i) the first two bulbs are good and the third is defective;

(ii) exactly one of the three bulbs is defective.

Answer (3 marks)

Good bulbs = 5, defective = 3, total = 8. Let G_i, D_i denote good/defective at the i-th draw.

(i) By the multiplication theorem,

$P(G_1G_2D_3) = P(G_1) \cdot P(G_2 \mid G_1) \cdot P(D_3 \mid G_1G_2) = (5/8)(4/7)(3/6) = 60/336 = \mathbf{5/28}$.

(ii) Exactly one defective can happen as DGG, GDG or GGD.

$P(D_1G_2G_3) = (3/8)(5/7)(4/6) = 5/28$; $P(G_1D_2G_3) = (5/8)(3/7)(4/6) = 5/28$; $P(G_1G_2D_3) = 5/28$.

Required probability $= 3 \times 5/28 = \mathbf{15/28}$.

Marking: (i) Conditional probabilities and multiplication theorem – 1 mark · (i) Answer 5/28 – ½ mark · (ii) Three mutually exclusive cases – 1 mark · (ii) Answer 15/28 – ½ mark

Examiner tip: Without replacement the denominators drop (8, 7, 6). In (ii) do not forget that the defective bulb can come at any of the three draws.

30. Solve the following linear programming problem graphically:
Maximise $Z = 4x + 5y$
subject to the constraints $x + 3y \le 15$, $2x + y \le 10$, $x \ge 0$, $y \ge 0$.

Answer (3 marks)

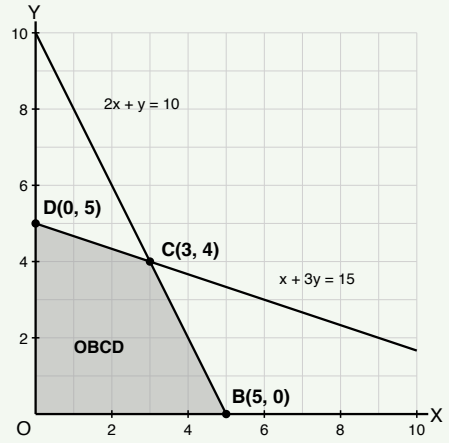
Draw the lines $x + 3y = 15$ (through (15, 0) and (0, 5)) and $2x + y = 10$ (through (5, 0) and (0, 10)). (0, 0) satisfies both inequalities, so each region is on the origin side. With $x \ge 0$, $y \ge 0$, the feasible region is the bounded quadrilateral OBCD (shaded).

C: $x + 3y = 15$ and $2x + y = 10 \Rightarrow y = 10 - 2x \Rightarrow x + 30 - 6x = 15 \Rightarrow x = 3$, $y = 4$. So C(3, 4).

Corner point	Z = 4x + 5y
O(0, 0)	0
B(5, 0)	20
C(3, 4)	32
D(0, 5)	25

The feasible region is bounded, so the maximum occurs at a corner point.

Maximum Z = 32 at x = 3, y = 4.



Marking: Correct graph and shaded feasible region – 1 mark · Corner points including C(3, 4) – 1 mark · Table and maximum Z = 32 at (3, 4) – 1 mark

Examiner tip: Label the corner points on the graph and evaluate Z at every corner; mention that the region is bounded.

31. (A) If $y = (\tan x)^x + (x^2 + 1)^{\log x}$, $0 < x < \pi/2$, find dy/dx.

OR

(B) If $x = t + 1/t$ and $y = t - 1/t$, find d^2y/dx^2 at $t = 2$.

Answer (3 marks)

(A) Let $u = (\tan x)^x$ and $v = (x^2 + 1)^{\log x}$, so $y = u + v$ and $dy/dx = du/dx + dv/dx$.

$\log u = x \log(\tan x) \Rightarrow (1/u)\, du/dx = \log(\tan x) + x \cdot \sec^2 x/\tan x = \log(\tan x) + 2x/\sin 2x$.

$\Rightarrow du/dx = (\tan x)^x [\log(\tan x) + 2x/\sin 2x]$.

$\log v = (\log x) \cdot \log(x^2 + 1) \Rightarrow (1/v)\, dv/dx = (1/x) \log(x^2 + 1) + (\log x) \cdot 2x/(x^2 + 1)$.

$\Rightarrow dv/dx = (x^2 + 1)^{\log x} [\log(x^2 + 1)/x + 2x \log x/(x^2 + 1)]$.

$dy/dx = (\tan x)^x [\log(\tan x) + 2x/\sin 2x] + (x^2 + 1)^{\log x} [\log(x^2 + 1)/x + 2x \log x/(x^2 + 1)]$.

($\sec^2 x/\tan x = 1/(\sin x \cos x) = 2/\sin 2x$; leaving it as $x \sec^2 x/\tan x$ is also correct.)

OR

(B) $dx/dt = 1 - 1/t^2 = (t^2 - 1)/t^2$, $dy/dt = 1 + 1/t^2 = (t^2 + 1)/t^2$.

$dy/dx = (dy/dt)/(dx/dt) = (t^2 + 1)/(t^2 - 1)$.

$d/dt\,(dy/dx) = [2t(t^2 - 1) - 2t(t^2 + 1)]/(t^2 - 1)^2 = -4t/(t^2 - 1)^2$.

$d^2y/dx^2 = d/dt\,(dy/dx) \times dt/dx = [-4t/(t^2 - 1)^2] \times [t^2/(t^2 - 1)] = -4t^3/(t^2 - 1)^3$.

At $t = 2$: $d^2y/dx^2 = -4(8)/(3)^3 =$ **$-32/27$**.

Marking: (A) du/dx by logarithmic differentiation – 1½ marks · dv/dx – 1 mark · Final dy/dx – ½ mark · (B) $dy/dx = (t^2 + 1)/(t^2 - 1)$ – 1 mark · d/dt(dy/dx) and multiplying by dt/dx – 1 mark · Value −32/27 at t = 2 – 1 mark

Examiner tip: (A) Differentiate u and v separately – never take log of a sum. (B) d^2y/dx^2 is NOT $(d^2y/dt^2)/(d^2x/dt^2)$; multiply d/dt(dy/dx) by dt/dx.

SECTION D This section comprises 4 long answer (LA) type questions of 5 marks each. **[20]**

32. If $A = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 1 & -2 \\ -1 & 2 & 3 \end{bmatrix}$, find A^{-1}. Hence, solve the following system of equations:

$x + 2y + 2z = 1$, $3x + y - 2z = 5$, $-x + 2y + 3z = -4$.

Answer (5 marks)

$|A| = 1(3 + 4) - 2(9 - 2) + 2(6 + 1) = 7 - 14 + 14 = 7 \neq 0$, so A^{-1} exists.

Cofactors:

$A_{11} = 7$, $A_{12} = -(9 - 2) = -7$, $A_{13} = 6 + 1 = 7$;

$A_{21} = -(6 - 4) = -2$, $A_{22} = 3 + 2 = 5$, $A_{23} = -(2 + 2) = -4$;

$A_{31} = -4 - 2 = -6$, $A_{32} = -(-2 - 6) = 8$, $A_{33} = 1 - 6 = -5$.

$$\text{adj } A = \begin{bmatrix} 7 & -2 & -6 \\ -7 & 5 & 8 \\ 7 & -4 & -5 \end{bmatrix}$$

$$A^{-1} = (1/|A|)\,\text{adj } A = (1/7)\begin{bmatrix} 7 & -2 & -6 \\ -7 & 5 & 8 \\ 7 & -4 & -5 \end{bmatrix}$$

The system is $AX = B$ with $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 5 \\ -4 \end{bmatrix}$.

$$X = A^{-1}B = (1/7)\begin{bmatrix} 7 - 10 + 24 \\ -7 + 25 - 32 \\ 7 - 20 + 20 \end{bmatrix} = (1/7)\begin{bmatrix} 21 \\ -14 \\ 7 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

$x = 3$, $y = -2$, $z = 1$.

Check: $3 - 4 + 2 = 1$ ✓; $9 - 2 - 2 = 5$ ✓; $-3 - 4 + 3 = -4$ ✓.

Marking: $|A| = 7$ – 1 mark · Cofactors and adj A – 2 marks · A^{-1} – ½ mark · Writing $AX = B$ and $X = A^{-1}B$ – ½ mark · $x = 3$, $y = -2$, $z = 1$ – 1 mark

Examiner tip: adj A is the TRANSPOSE of the cofactor matrix. Here the coefficient matrix of the system is A itself, so $X = A^{-1}B$ (not $(A')^{-1}B$). Substitute back to check.

33. (A) Show that the relation R on the set Z of integers, given by $R = \{(a, b) : a^2 - b^2$ is divisible by 3$\}$, is an equivalence relation. Write the equivalence classes [0] and [1], and show that there are exactly two distinct equivalence classes.

OR

(B) Show that the function $f : R - \{2\} \to R - \{3\}$ defined by $f(x) = (3x + 1)/(x - 2)$ is one-one and onto. Also examine whether the function $g : R \to R$ defined by $g(x) = 3x^2 + 1$ is one-one or onto.

Answer (5 marks)

(A) **Reflexive:** for every $a \in Z$, $a^2 - a^2 = 0 = 3 \times 0$, so $(a, a) \in R$.

Symmetric: $(a, b) \in R \Rightarrow a^2 - b^2 = 3m$ for some $m \in Z \Rightarrow b^2 - a^2 = 3(-m) \Rightarrow (b, a) \in R$.

Transitive: $(a, b), (b, c) \in R \Rightarrow a^2 - b^2 = 3m$, $b^2 - c^2 = 3n \Rightarrow a^2 - c^2 = 3(m + n) \Rightarrow (a, c) \in R$.
Hence R is an equivalence relation.

[0] $= \{b \in Z : 0 - b^2$ is divisible by $3\} = \{b : 3$ divides $b^2\} = \{b : 3$ divides $b\}$ (as 3 is prime) $= \{..., -6, -3, 0, 3, 6, ...\}$.

[1] $= \{b \in Z : 1 - b^2$ is divisible by $3\}$. Every integer not divisible by 3 is of the form $3q \pm 1$, and $(3q \pm 1)^2 = 3(3q^2 \pm 2q) + 1$, so $1 - b^2$ is divisible by 3. If 3 divides b, $1 - b^2$ is not divisible by 3. So $[1] = \{b : b$ is not a multiple of $3\} = \{..., -4, -2, -1, 1, 2, 4, 5, ...\}$.
$[0] \cup [1] = Z$ and $[0] \cap [1] = \phi$, so every integer lies in one of these two classes: there are **exactly two distinct equivalence classes**.

OR

(B) **One-one:** let $x_1, x_2 \in R - \{2\}$ with $f(x_1) = f(x_2)$.

$(3x_1 + 1)(x_2 - 2) = (3x_2 + 1)(x_1 - 2)$

$\Rightarrow 3x_1x_2 - 6x_1 + x_2 - 2 = 3x_1x_2 - 6x_2 + x_1 - 2 \Rightarrow 7x_2 = 7x_1 \Rightarrow x_1 = x_2$. So f is one-one.

Onto: let $y \in R - \{3\}$. Solve $y = (3x + 1)/(x - 2)$: $xy - 2y = 3x + 1 \Rightarrow x = (2y + 1)/(y - 3)$.
This x is a real number since $y \neq 3$, and $x \neq 2$ ($x = 2$ would give $2y + 1 = 2y - 6$, impossible). Also $f(x) = [3(2y + 1)/(y - 3) + 1]/[(2y + 1)/(y - 3) - 2] = (7y)/7 = y$. So every y has a pre-image: f is onto.

$g(x) = 3x^2 + 1$: $g(1) = g(-1) = 4$, so g is not one-one. $g(x) \geq 1$ for all x, so $0 \in R$ has no pre-image: g is not onto. Hence g is **neither one-one nor onto**.

Marking: (A) Reflexive – 1 mark · Symmetric – 1 mark · Transitive – 1 mark · [0] = multiples of 3 – 1 mark · [1] = non-multiples of 3 and exactly two classes – 1 mark · (B) f one-one – 2 marks · f onto (pre-image in the domain, $x \neq 2$) – 2 marks · g neither one-one nor onto with counter-examples – 1 mark

Examiner tip: (A) To prove each property, start from the hypothesis in symbols and reach the conclusion; examples are not proofs. (B) For onto, show that the pre-image lies in the domain ($x \neq 2$) – this step is often missed.

34. (A) Using integration, find the area of the region bounded by the parabola $y = x^2$, the line $x + y = 6$ and the x-axis.

OR

(B) Using integration, find the area of the smaller region into which the ellipse $x^2/36 + y^2/16 = 1$ is divided by the line $x = 3$.

Answer (5 marks)

(A) Intersection of $y = x^2$ and $x + y = 6$ (in the first quadrant): $x^2 + x - 6 = 0 \Rightarrow (x + 3)(x - 2) = 0 \Rightarrow x = 2$ ($x = -3$ gives a point not on the boundary of this region). So P(2, 4).
The line meets the x-axis at B(6, 0). The region (shaded) is bounded above by the parabola from $x = 0$ to 2 and by the line from $x = 2$ to 6.

Area $= \int_0^2 x^2\,dx + \int_2^6 (6 - x)\,dx$

$= [x^3/3]_0^2 + [6x - x^2/2]_2^6$

$= 8/3 + [(36 - 18) - (12 - 2)] = 8/3 + 8$

$=$ **32/3 square units**.

OR

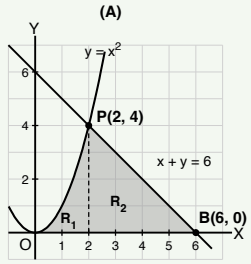
(B) On the ellipse, $y = \pm(2/3)\sqrt{(36 - x^2)}$. The line $x = 3$ meets the ellipse where $y^2 = 16(1 - 9/36) = 12$, i.e. at $P(3, 2\sqrt{3})$ and $Q(3, -2\sqrt{3})$. The smaller region lies between $x = 3$ and $x = 6$ (shaded) and is symmetric about the x-axis.

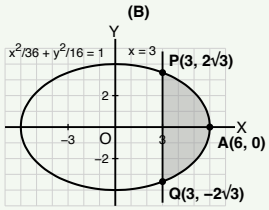
Area $= 2 \int_3^6 (2/3)\sqrt{(36 - x^2)}\,dx = (4/3)\,[\,(x/2)\sqrt{(36 - x^2)} + (36/2)\sin^{-1}(x/6)\,]_3^6$

$= (4/3)\,[\,(0 + 18 \cdot \pi/2) - ((3/2) \cdot 3\sqrt{3} + 18 \cdot \pi/6)\,]$

= (4/3) [9π − 9√3/2 − 3π] = (4/3) [6π − 9√3/2]

= **(8π − 6√3) square units**.





Marking: (A) Sketch of the region – 1 mark · Point of intersection P(2, 4) and B(6, 0) – 1 mark · Correct set-up of integrals – 1 mark · Evaluation of both integrals – 1 mark · Area 32/3 sq. units – 1 mark · (B) Sketch and points of intersection P(3, 2√3), Q(3, −2√3) – 1 mark · Using symmetry, area = $2\int_3^6 (2/3)\sqrt{(36 - x^2)}\,dx$ – 1 mark · Correct standard formula – 1 mark · Substituting limits – 1 mark · Area (8π − 6√3) sq. units – 1 mark

Examiner tip: Draw the sketch first: in (A) the upper boundary changes at x = 2, so the area must be split into two integrals. In (B), on the ellipse $y = (4/6)\sqrt{(36 - x^2)}$: do not drop the factor 2/3. Write 'square units' with the answer.

35. A line l_1 passes through the points A(−1, 2, 0) and B(1, 5, 2). Another line l_2 is given by (x − 4)/2 = (y + 1)/2 = (z − 2)/1.

(i) Write the vector equations of l_1 and l_2.

(ii) Find the shortest distance between l_1 and l_2.

(iii) Do the lines intersect? Justify your answer.

Answer (5 marks)

(i) l_1 is parallel to the vector from A to B: $(1 + 1)\hat{i} + (5 - 2)\hat{j} + (2 - 0)\hat{k} = 2\hat{i} + 3\hat{j} + 2\hat{k}$.

l_1: $\vec{r} = (-\hat{i} + 2\hat{j}) + \lambda(2\hat{i} + 3\hat{j} + 2\hat{k})$

l_2: $\vec{r} = (4\hat{i} - \hat{j} + 2\hat{k}) + \mu(2\hat{i} + 2\hat{j} + \hat{k})$

(ii) Here $\vec{a}_1 = -\hat{i} + 2\hat{j}$, $\vec{b}_1 = 2\hat{i} + 3\hat{j} + 2\hat{k}$, $\vec{a}_2 = 4\hat{i} - \hat{j} + 2\hat{k}$, $\vec{b}_2 = 2\hat{i} + 2\hat{j} + \hat{k}$.

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 2 \\ 2 & 2 & 1 \end{vmatrix} = \hat{i}(3 - 4) - \hat{j}(2 - 4) + \hat{k}(4 - 6) = -\hat{i} + 2\hat{j} - 2\hat{k};\ |\vec{b}_1 \times \vec{b}_2| = \sqrt{(1 + 4 + 4)} = 3.$$

$\vec{a}_2 - \vec{a}_1 = 5\hat{i} - 3\hat{j} + 2\hat{k}$.

$(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2) = -5 - 6 - 4 = -15$.

Shortest distance $d = |(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2)| \,/\, |\vec{b}_1 \times \vec{b}_2| = 15/3 =$ **5 units**.

(iii) Direction ratios 2, 3, 2 and 2, 2, 1 are not proportional, so the lines are not parallel. Since the shortest distance is 5 ≠ 0, the lines **do not intersect**; they are skew lines.

Marking: (i) Vector equations of both lines – 1 mark · (ii) $\vec{b}_1 \times \vec{b}_2$ and its magnitude – 1 mark · $\vec{a}_2 - \vec{a}_1$ and the scalar triple product – 1 mark · Shortest distance = 5 units – 1 mark · (iii) Not parallel and d ≠ 0 ⇒ skew, do not intersect – 1 mark

Examiner tip: Take the modulus in the formula: the dot product is −15 but a distance is never negative.

SECTION E This section comprises 3 case study based questions of 4 marks each. Questions 36 and 37 have three sub-parts of 1, 1 and 2 marks; Question 38 has two sub-parts of 2 marks each. **[12]**

36. **Case study:** A district milk co-operative in western Maharashtra collects milk from three villages P, Q and R. Village P supplies 40%, village Q supplies 35% and village R supplies 25% of the milk cans. Laboratory records show that 3% of the cans from P, 2% of the cans from Q and 4% of the cans from R fail the quality test. A can is selected at random from the day's collection.

Based on the given information, answer the following questions:

(i) Find the probability that the selected can is from village Q and fails the quality test. [1]

(ii) Find the probability that the selected can fails the quality test. [1]

(iii) (a) If the selected can fails the quality test, find the probability that it came from village R. [2]

OR

(iii) (b) If the selected can passes the quality test, find the probability that it came from village P. [2]

Answer (4 marks)

Let E_1, E_2, E_3 be the events that the can is from P, Q, R and A the event that it fails the test.

$P(E_1) = 0.40$, $P(E_2) = 0.35$, $P(E_3) = 0.25$; $P(A \mid E_1) = 0.03$, $P(A \mid E_2) = 0.02$, $P(A \mid E_3) = 0.04$.

(i) $P(E_2 \cap A) = P(E_2)\,P(A \mid E_2) = 0.35 \times 0.02 =$ **0.007**.

(ii) By the theorem of total probability,

$P(A) = 0.40 \times 0.03 + 0.35 \times 0.02 + 0.25 \times 0.04 = 0.012 + 0.007 + 0.010 =$ **0.029**.

(iii) (a) By Bayes' theorem, $P(E_3 \mid A) = P(E_3)\,P(A \mid E_3)/P(A) = 0.010/0.029 =$ **10/29**.

OR

(iii) (b) $P(A') = 1 - 0.029 = 0.971$ and $P(A' \mid E_1) = 1 - 0.03 = 0.97$.

$P(E_1 \mid A') = P(E_1)\,P(A' \mid E_1)/P(A') = (0.40 \times 0.97)/0.971 = 0.388/0.971 =$ **388/971**.

Marking: (i) 0.007 – 1 mark · (ii) Total probability 0.029 – 1 mark · (iii)(a) Bayes' theorem set-up – 1 mark, answer 10/29 – 1 mark · (iii)(b) $P(A')$ and $P(A' \mid E_1)$ – 1 mark, answer 388/971 – 1 mark

Examiner tip: Write the events in symbols first. In (iii)(b) use the probabilities of PASSING (0.97 and 0.971), not of failing.

37. **Case study:** During a crop survey near Baramati, the height of a drone above the ground (in metres), t minutes after the start of the recording, is given by $h(t) = t^3 - 9t^2 + 24t + 2$, $0 \le t \le 5$.

Based on the given information, answer the following questions:

(i) Find the values of t at which $h'(t) = 0$. [1]

(ii) Find the intervals in which the drone is rising (h strictly increasing) and in which it is coming down (h strictly decreasing). [1]

(iii) (a) Using the second derivative test, find the local maximum and local minimum heights of the drone. [2]

OR

(iii) (b) Find the greatest and the least heights of the drone during $0 \le t \le 5$. [2]

Answer (4 marks)

(i) $h'(t) = 3t^2 - 18t + 24 = 3(t^2 - 6t + 8) = 3(t - 2)(t - 4)$. $h'(t) = 0 \Rightarrow$ **t = 2 or t = 4**.

(ii) For $0 < t < 2$: $h'(t) > 0$; for $2 < t < 4$: $h'(t) < 0$; for $4 < t < 5$: $h'(t) > 0$.

The drone is **rising in (0, 2) and (4, 5)** and **coming down in (2, 4)**.

(iii) (a) $h''(t) = 6t - 18$.

$h''(2) = -6 < 0 \Rightarrow$ local maximum at t = 2: $h(2) = 8 - 36 + 48 + 2 =$ **22 m**.

$h''(4) = 6 > 0 \Rightarrow$ local minimum at t = 4: $h(4) = 64 - 144 + 96 + 2 =$ **18 m**.

OR

(iii) (b) Values at the critical points and the end points: $h(0) = 2$, $h(2) = 22$, $h(4) = 18$, $h(5) = 125 -$

225 + 120 + 2 = 22.

Greatest height = 22 m (at t = 2 min and again at t = 5 min); **least height = 2 m** (at t = 0).

Marking: (i) t = 2, 4 – 1 mark · (ii) Correct intervals – 1 mark · (iii)(a) h″ test at both points – 1 mark, heights 22 m and 18 m – 1 mark · (iii)(b) Values at t = 0, 2, 4, 5 – 1 mark, greatest 22 m and least 2 m – 1 mark

Examiner tip: For greatest/least values on a closed interval, also check the END points: here h(5) equals the local maximum and h(0) is the least value.

38. **Case study:** A triangular shade sail is fixed over a school courtyard. Taking one corner of the courtyard as the origin and distances in metres, the three corners of the sail are at A(1, 0, 2), B(3, 2, 3) and C(3, −1, 0).

Based on the given information, answer the following questions:

(i) Taking $\vec{a}$ as the vector from A to B and $\vec{b}$ as the vector from A to C, find $\vec{a} \times \vec{b}$ and hence the area of the sail. [2]

(ii) Show that $\angle BAC = 90°$ and find a unit vector perpendicular to the plane of the sail. [2]

Answer (4 marks)

(i) $\vec{a} = (3-1)\hat{i} + (2-0)\hat{j} + (3-2)\hat{k} = 2\hat{i} + 2\hat{j} + \hat{k}$; $\vec{b} = (3-1)\hat{i} + (-1-0)\hat{j} + (0-2)\hat{k} = 2\hat{i} - \hat{j} - 2\hat{k}$.

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 2 & 1 \\ 2 & -1 & -2 \end{vmatrix} = \hat{i}(-4+1) - \hat{j}(-4-2) + \hat{k}(-2-4) = \mathbf{-3\hat{i} + 6\hat{j} - 6\hat{k}}.$$

$|\vec{a} \times \vec{b}| = \sqrt{(9 + 36 + 36)} = 9$. Area of the sail = ½ $|\vec{a} \times \vec{b}|$ = ½ × 9 = **9/2 m^2 (4.5 m^2)**.

(ii) $\vec{a} \cdot \vec{b} = 4 - 2 - 2 = 0$, so $\vec{a} \perp \vec{b}$, i.e. AB ⊥ AC and **$\angle BAC = 90°$**.

$\vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$, hence to the plane of the sail. Unit vector = $(-3\hat{i} + 6\hat{j} - 6\hat{k})/9$ = **$\pm(1/3)(-\hat{i} + 2\hat{j} - 2\hat{k})$**.

Marking: (i) $\vec{a}$ and $\vec{b}$ – ½ mark · Cross product – 1 mark · Area 9/2 m^2 – ½ mark · (ii) $\vec{a} \cdot \vec{b} = 0 \Rightarrow \angle BAC = 90°$ – 1 mark · Unit vector $\pm(-\hat{i} + 2\hat{j} - 2\hat{k})/3$ – 1 mark

Examiner tip: Area of a triangle is ½$|\vec{a} \times \vec{b}|$, not $|\vec{a} \times \vec{b}|$. A unit vector perpendicular to a plane can point either way, so write ±.

All the best for your exam!